THE MASER AS A PARAMETRIC AMPLIFIER

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MANY PEOPLE have noted an amusing similarity between the maser and the parametric amplifier, whose behavior is also governed by "pseudo-quantum" laws, such as the Manley-Rowe equations, reminiscent of $E = \hbar\omega$. Relations giving a proportionality between energy and frequency have a long history in physics, and are characteristic of many purely classical systems. The best-known example is the adiabatic theorem, which played an important role in the early development of quantum theory. One finds that in any classical periodic system, the action integral over one period is an approximate invariant under slowly varying perturbations. The derivation of this law is particularly simple in the case of a harmonic oscillator with slowly varying spring constant. Here the oscillator coordinate satisfies the equation of motion $\ddot{x} + \omega^2 x = 0$. If now we allow ω to be a slowly varying function of time, the BWK approximation to the solution is

$$x(t) = \frac{1}{\sqrt{\omega}} \exp[i \int \omega(t) dt] \tag{1}$$

and so the energy is

$$E = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \omega^2 x^2 = (\text{const.})\ (\omega). \tag{2}$$

This adiabatic theorem has recently found several applications, ranging from a simple derivation of the Slater perturbation formula in microwave theory to the calculation of orbits in particle accelerators.

Now consider a quantum-mechanical system such as a molecule in a maser, interacting with an electromagnetic field. We

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wish to show that the Schrödinger equation describing the time-evolution of this system has a close mathematical analogy to classical parametric systems, and in fact discloses a particular form of classical Hamiltonian for which the action conservation law becomes rigorous, independently of the magnitude or rate of change of the perturbations. Let the stationary state vectors of the quantum-mechanical system be u_n for the energy levels $E_n = \hbar\omega_n$, and expand the time-dependent wave function in the usual way,

$$\psi(t) = \sum_n a_n(t)\, u_n. \tag{3}$$

The equations of motion are then

$$i\hbar\dot{a}_m = \sum_n H_{mn} a_n = E_m a_m + \sum_n V_{mn}(t)\, a_n \tag{4}$$

where $V_{mn}(t)$ are the matrix elements of the interaction with fields, for example the product of dipole moment operator with electric field $E(t)$. By introducing the quadratic form which represents the expectation value of the energy,

$$H = \sum_{mn} \mathrm{H}_{mn}\, a_m^*\, a_n \tag{5}$$

we can write the equations of motion in a form resembling the classical Hamiltonian equations:

$$i\hbar\dot{a}_m = \frac{\partial H}{\partial a_m^*}, \qquad i\hbar\dot{a}_m^* = -\frac{\partial H}{\partial a_m}. \tag{6}$$

To increase the resemblance, we introduce the real quantities $p_n(t)$, $q_n(t)$ defined by

$$a_n = \frac{p_n - i\omega_n q_n}{(2\hbar\omega_n)^{\frac{1}{2}}}. \tag{7}$$

In terms of them, the quantity (5) becomes

$$H(q,p) = \frac{1}{2}\sum_n (p_n^2 + \omega_n^2\, q_n^2)$$

$$+ \frac{1}{2}\sum_{mn} [a_{mn}(p_m p_n + \omega_m\omega_n\, q_m q_n) + 2b_{mn}\omega_m\, q_m p_n] \tag{8}$$

where $a_{mn}(t)$, $b_{mn}(t)$ are proportional to the real and imaginary parts of V_{mn}, and the Equations of motion (4) and (6) reduce to

$$\dot{q}_m = \frac{\partial H}{\partial p_m}, \qquad \dot{p}_m = -\frac{\partial H}{\partial q_m}. \tag{9}$$

Equations (8) and (9) are, of course, nothing but the Schrödinger equation, in unconventional notation.

In consequence of the fact that H_{mn} is Hermitian, the Equation of motion (4) has a rigorous constant of the motion

$$\sum_m |a_m|^2 = \text{constant} \tag{10}$$

which in quantum theory we interpret as "conservation of probability." Using Equation (7), we find that in terms of p_n, q_n this conservation law becomes

$$\sum_n \frac{p_n^2 + \omega_n^2 q_n^2}{2\omega_n} = \sum_n \frac{W_n}{\omega_n} = \text{constant} \tag{11}$$

where W_n is the energy stored in the n'th mode. Equation (11) is also easily verified directly from Equations (8) and (9).

If we had been shown only the final Equations (8), (9), and (11), and not the argument which I have used to derive them here, a very different interpretation would seem natural. In Equations (8) and (9) we have an assemblage of classical harmonic oscillators perturbed by some external environment in a manner described by the matrices $a_{mn}(t)$, $b_{mn}(t)$. Since the Hamiltonian (8) is quadratic in the p_n, q_n, for any particular values of the a_{mn}, b_{mn} we could find a new set of normal modes; the effect of the environment is to vary the spring constants. The set of harmonic oscillators is not coupled directly to its environment, but parametrically. Thus to every kind of level scheme which one might use in a maser, there corresponds a purely classical parametric system which would behave in just the same manner and what is most important, would react back on the perturbing environment in the same way as does the atom or molecule.

As a consequence of this analogy, the decision whether a maser or a parametric amplifier is best for any given application

might involve the following reasoning. For many jobs which a maser can do, we can in principle find a classical parametric system which would do the same job. So it must be the practical considerations, such as availability of materials with certain relaxation times, stability of parameters, efficient parametric circuit elements, etc., which lead us to prefer one kind of device to another. Many years ago, W. W. Hansen proposed a theorem which may or may not apply to this case; given two different ways of accomplishing something, both of which will work in principle, that one will be best which receives the greatest number of man-hours of development work.

DISCUSSION

I. R. SENITZKY: I would like to make two comments.

1. The spontaneous emission properties of the molecule have not been completely considered, since the field has not been quantized. Thus, a molecule may be regarded as a classical parametric system only if some of the quantum-mechanical properties are neglected.

2. It seems that the essential difference between a maser and a parametric amplifier (ignoring now the quantum-mechanical aspects mentioned above) is that a maser is a collection of many loosely-coupled systems, while a parametric amplifier is a single system. Thus, there is negligible correlation between the idler oscillations of the many molecules of the maser, while the parametric amplifier has a single idler oscillation. Any effects, therefore, which are due to idler oscillation (such as the ones mentioned in my talk yesterday) will be entirely different in a maser than in a parametric amplifier.

E. T. JAYNES: 1. Surely. When we write $E(t)$, we are implying semiclassical radiation theory. However, as I showed in a report last year, this theory does give spontaneous emission, with the correct Einstein A-coefficients, if we take the expectation value of dipole moment as the source for a classical electromagnetic field. Field quantization is not necessary for spontaneous emission.

2. The "idler oscillations" involve the absolute phase of the wave function $\psi(t)$, which is not observable. In the classical Hamiltonian, [Equation (8)], this corresponds to the fact that the interaction term involves coordinates and momenta in a form which contains only the difference frequencies $\omega_m - \omega_n$, not the sums $\omega_m + \omega_n$.

M. WEISS: In addition to deriving the Manley-Rowe relations by means of conservation of the number of quanta, it is also possible to derive the Tien phase relations for a traveling wave parametric amplifier by requiring the conservation of momentum. Thus,

$$\left(\frac{h\nu}{v\varphi}\right)_{\text{pump}} = \left(\frac{h\nu}{v\varphi}\right)_{\text{signal}} + \left(\frac{h\nu}{v\varphi}\right)_{\text{idler}}$$

results in

$$\beta_{\text{signal}} + \beta_{\text{idler}} = \beta_{\text{pump}}.$$

It is to be noted that for momentum in a dispersive medium one must use the phase velocity. This quantum analog is particularly useful in the derivation of the phase relations of more than three frequency traveling wave parametric amplifiers.

G. GOULD: Professor Townes has mentioned an early electron maser, the triode. The Barkhauser-Kurz oscillator is more easily understood. Electrons oscillate approximately harmonically in a one-dimensional potential well between plate and cathode. The electrons are injected into a band of levels whose vibrational quantum numbers are $n \cong [E/h\nu] - \frac{1}{2} \sim 10^8$. An oscillating electric field induces transitions to empty lower and higher levels, depending on phase. Those which absorb power are removed, leaving a net induced emission of photons to the e.m. field.

Similarly, in the magnetron type of maser, stimulated emission of radiation takes place as the electrons undergo transitions to states of lower angular momentum quantum number.

M. W. P. STRANDBERG: An analytic definition which seems to make physical sense has been given by Strandberg [Phys. Rev. (1958), "Spin-Lattice Relaxation"]. A solid state maser or pa-

ramagnetic amplifier by this definition is one in which the pumping field interacts with a system which has such a short phase memory that only the diagonal elements of the density matrix are affected, for example, a system with $T_2 \ll T_1$. A parametric amplifier is one such that the pumping field is able to impose phase coherence on the system, so that both diagonal and off-diagonal elements of the density matrix are affected.