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Long-Term Solutions in Semiclassical Radiation Theory*

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An improved form of semiclassical radiation theory is developed which includes the effect of the atom's radiation field back on the atom. This formalism is applied to the problem of a single "two-level atom" interacting with a monochromatic field. The resulting equations are solved without resorting to time-dependent perturbation theory, and are found to predict the behavior of the system over times long compared with the lifetime for spontaneous transitions. Not only stimulated emission and absorption, but also spontaneous emission with the proper Einstein A coefficient, and a frequency shift which agrees at least semiquantitatively with the Lamb shift are described. In addition, several nonlinear effects involving the interference between spontaneous and stimulated radiation are described, and new experiments which might detect such effects are suggested.

I. INTRODUCTION

The theory of interaction of optical radiation with atoms, beginning with Einstein's introduction of the A and B coefficients[1] and refined by later applications of quantum theory,[2-5] has for many years predicted all aspects of these phenomena on which we have experimental evidence. However, this evidence, while large in volume, is limited in scope. Recent advances (lasers, etc.) bring into the area of feasible experiments a wider range of phenomena, in which our present theory has received almost no experimental tests thus far. In planning new significant experiments, a more detailed theoretical treatment of these phenomena is needed, part of which is given in the present work.

The original optical experiments (from roughly 1880 to 1930) provided most of the clues on which our present quantum theory is based, but, in fact, they gave evidence only on what might be called the "kinematic" and "amplitude" aspects, and not on the "dynamic" or the coherence aspects, of radiation phenomena. For example, they gave quantitative verifications of the positions of spectral lines, but not about such dynamic matters as the shape of the spontaneous emission pulse emitted by an atom, or about the mutual coherence of radiation from nearby atoms. Absorption spectrum experiments also gave line positions, but no evidence about the dynamics by which the absorption develops in time with sudden illumination. Likewise, one observed the amplitude and polarization of resonance radiation, but not its degree of phase coherence with the primary radiation, or the dynamics of its transient buildup when an atom is suddenly illuminated.

More recent experiments have continued to check the kinematic rather than the dynamical aspects of radiation theory. Thus, the existing experiments on the Lamb shift[6-9] have measured only the incident frequency needed to initiate a transition, starting from a metastable S state. The experimental fact is that upward transitions are observed to start at a lower frequency, and downward ones at a higher frequency, than predicted by the elementary solution of the Dirac equation. This is consistent with the view that the Lamb shift arises from a raising of the S levels relative to others; but it is equally consistent with the view that radiative line shifts increase the frequency of all downward transitions and lower the frequency of upward ones, whose behavior could not be described in terms of level shifts at all. Nor do the existing experiments answer the question whether the radiative line shift remains constant throughout the transition; they are equally consistent with a theory (which, as we will see, has some *raison d'être*) in which radiative effects raise the frequency of all lines during the part of the emission or absorption when the atom is near the upper level, and lower it when the atom is near the lower level. Further experiments are thus needed before one could claim that our present theory of the Lamb shift is established to the exclusion of thinkable alternatives.

Thus far, dynamical and coherence properties have been observed only as smeared-out statistical averages, which do not permit inferences about the underlying elementary processes. Thus, existing experiments on blackbody radiation mea-

sure only its spectral density, and not further details of the field fluctuations. Such measurements are consistent with the notion of sudden exchanges of energy $\hbar\omega$ between field and atom; but they do not require this, since a continuous energy exchange following any curve of a certain symmetry will lead to just the same time average for the field energy in any spectral region. Likewise, the observed exponential decay of fluorescence is consistent with the idea that each atom emits an exponentially damped wave train, as in the Wigner-Weisskopf theory[4]; but it does not require this, since the total radiation will fall off exponentially, whatever the shape of the basic spontaneous emission pulse, provided only that the number of emitting atoms falls off exponentially.

There are other dynamical effects on which we can hardly claim to have experimental evidence, even in the form of statistical averages. For example, what interference effects are to be expected when spontaneous and induced emission proceed simultaneously? What nonlinear effects result when an atom interacts with a coherent radiation field over long time intervals? The state of such an atom can always be described by a density matrix $\rho(t)$. Does $\rho(t)$ settle down to a steady state, approach a periodic limit cycle, or exhibit some more complicated long-time behavior? All of the aforementioned dynamical and coherence effects appear to be in the range where more detailed experimental evidence could now be obtained.

As far as the theoretical treatment of these phenomena is concerned, the distinction between "dynamic" and "kinematic" is simply whether time-dependent perturbation theory is sufficient to describe the phenomena. To describe "dynamic" phenomena one must develop means of solving the Schrödinger equation accurately over long time intervals (e.g., in which an atom undergoes a large change of state), but without losing phase information. The purpose of this paper is to develop such a method of solution, and to present several applications, leading to a number of predictions not yet tested by experiment but on which the experiments are now feasible.

2. QUANTUM ELECTRODYNAMICS AND SEMICLASSICAL THEORY

Further study of dynamical and coherence phenomena may prove to have a greater fundamental importance than would appear at first glance. It is well known that our present theory of radiation, quantum electrodynamics (QED), faces serious logical and mathematical difficulties.[10] The appearance of divergent integrals in almost every nontrivial problem can be traced in part to infinite vacuum fluctuations, and in part to the particular propagators used. The Feynman propagator $S_F(x-x')$ raises a difficulty about relativistic causality in that it fails to vanish for spacelike intervals $(x-x')$. In addition, it has a singularity $(c^2t^2-r^2)^{-2}$ on the light-cone,[11] which alone is sufficient to guarantee divergence of almost any integral of the form $\int S_F(x-x')\cdot f(x')d^4x'$. It is only after using devices such as regulators and renormalization techniques, or by striking out certain divergent expressions on grounds of Lorentz or gauge invariance, that one obtains finite results that can be compared with experiment.

Yet the finite parts of the propagator S_F appear necessary for agreement with experiment. For example, in the calculation of vacuum polarization given by Bjorken and Drell,[12] use of the retarded propagator, instead of S_F, greatly improves the convergence and removes the difficulty about causality; but it also leads to the prediction of zero vacuum polarization. It might be thought possible to separate S_F into its "physical" and "nonphysical" parts; but as yet no one has seen how to do this.

For these reasons, there has been a growing dissatisfaction with the present theory, in spite of its experimental successes. Even Dirac,[13] the founder of QED, has recently described its present form as "a stopgap, without any lasting future." It is clear that, at the very least, there is something seriously wrong in our present *formulation* of this theory; but we lack clues telling us in what specific way it should be modified.

Most recent study of these problems has tended to concentrate on the high-energy region. However, because of technological limitations, further high-energy experiments can only continue to give evidence on "amplitude" aspects of the theory. In view of recent advances, it is now possible to perform controlled experiments in the optical region – where duplication in the high energy region would be out of the question – which test aspects of QED on which we have as yet no direct evidence. For these reasons, we suggest that the missing clues are as likely to be found in the optical region as anywhere else, and so it is important that the optical phenomena be further explored.

In Sec. 3 we develop a method of treating the problem of an atom interacting with electromagnetic field without resorting to the usual time-dependent perturbation theory. The solutions will retain their phase and amplitude accuracy over long periods of time.

In Sec. 4 a number of detailed solutions are presented in which the effect of spontaneous emission and radiative line shifts on long-time behavior are exhibited.

In Sec. 5 we show that the method we have developed is quite general and could be applied just as well to any problem in radiation theory.

Finally, in Sec. 6 we discuss the differences between this method and the usual formalism. New experiments to test the detailed solutions of Sec. 4 are proposed.

3. EQUATIONS OF MOTION

Consider an atom with energy levels E_i and stationary-state wave functions ψi,

$$H_0\psi_i = E_i\psi_i \,. \tag{3.1}$$

In what follows we take into account only two of these levels, between which there is a dipole-moment matrix element μ. Let ω be the natural line frequency of this transition, and choose the zero from which we measure the energies to be midway between the two "active" levels, so that the energies are

$$E_2 = -E_1 = \tfrac{1}{2}\hbar\omega \,. \tag{3.2}$$

The state vector of the atom at an arbitrary time will be some linear combination of the active levels

$$\psi(t) = a(t)\psi_1 + b(t)\psi_2, \tag{3.3}$$

and the Schrödinger equation

$$i\hbar\dot{\psi} = [H_0 - \vec{\mu}\cdot\vec{\mathrm{E}}(t)]\psi$$

in which the dipole moment interacts with an electric field $E(t)$, then reduces to

$$\begin{aligned} i\hbar\dot{a}(t) &= E_1 a(t) - \vec{\mu}\cdot\vec{\mathrm{E}}(t)b(t), \\ i\hbar\dot{b}(t) &= E_2 b(t) - \vec{\mu}\cdot\vec{\mathrm{E}}(t)a(t). \end{aligned} \tag{3.4}$$

This neglect of all other levels is well justified if we suppose that ψ_1 is the ground state, so it cannot be depopulated by transitions to lower ones; and that the spectrum of the applied field $E(t)$ contains no appreciable component at any frequency which could couple ψ_1 and ψ_2 resonantly to other levels. Thus, no other levels can attain an appreciable amplitude: Their effect can be found by perturbation theory,[14] and amounts, for all practical purposes, to a small effective level shift, which we suppose to have been included already in E_1 and E_2,

We make one further assumption, that the field is linearly polarized with its polarization parallel to the atomic dipole moment. We will discuss more general applied fields in Sec. 4, and show that there are no qualitative changes in our results in that case. This assumption reduces Eq. (3.4) to scalar equations.

To better understand the physical influence of the field $E(t)$ on the atom, we introduce a new set of variables:

$$M(t) \equiv \langle \mu_{\mathrm{op}} \rangle = \mu(a^*b + ab^*), \tag{3.5}$$

and $$W(t) \equiv \langle H_0 \rangle = \tfrac{1}{2}\hbar\omega(|b|^2 - |a|^2). \tag{3.6}$$

As a result of Eqs. (3.4) these new variables satisfy

$$\frac{d^2}{dt^2}M + \omega^2 M = -K^2 W E(t), \tag{3.7a}$$

$$\dot{W} = \dot{M}E, \tag{3.7b}$$

where $K = 2\mu/\hbar$.

These equations are exactly equivalent to Eqs. (3.4), but in the form (3.7) we have a simple physical interpretation. The atomic dipole moment responds to an applied field $E(t)$ according to a driven harmonic-oscillator equation, with the unique feature that the coupling constant K^2W is proportional to the slowly varying energy W of the atom, reversing sign when W passes through zero. Thus, the dipole responds in opposite phase, depending on whether the atom is nearer to the upper state or the lower state. By this means, as we will presently see, the atom automatically adjusts its phase so as to give maximal induced emission when near the upper one. Equation (3.7b) is simply a statement of conservation of energy. These equations were derived previously to describe maser action.[15]

There is one further immediate consequence of writing the equations in this form: We have a first integral. Multiplying (3.7a) by $\dot{M}$ and substituting (3.7b), we get after integrating

$$\dot{M}^2 + \omega^2 M^2 + K^2 W^2 = \mu^2\omega^2 \,, \tag{3.8}$$

where we determined the constant of integration by substitution of the definitions of $M(t)$ and $W(t)$, and by use of the normalization condition

$$|a|^2 + |b|^2 = 1. \tag{3.9}$$

This first integral is of the form of the equation of a sphere. We can take advantage of this to define another convenient set of variables, which are just the Cartesian coordinates of a point on the unit sphere, defined by

$$\dot{M} + i\omega M \equiv i\mu\omega e^{i\Omega t}(x + iy), \tag{3.10a}$$

$$W \equiv \tfrac{1}{2}\hbar\omega z \,, \tag{3.10b}$$

where $x(t)$ and $y(t)$ are real, and Ω is the frequency of the applied field, which is supposed to be mono-

chromatic. Taking the imaginary part of (3.10),

$$M(t) = \mu(x \cos\Omega t - y \sin\Omega t), \tag{3.11}$$

we can see the physical meaning of these Cartesian coordinates. Thus, $x(t)$ is the component of the dipole moment in phase with the applied field, while $y(t)$ is the component 90° out of phase with the applied field, if the applied field is

$$E_{\text{app}} = E_0 \cos\Omega t. \tag{3.12}$$

The first integral (3.8) can be written in terms of these Cartesian variables giving

$$x^2 + y^2 + z^2 = 1, \tag{3.13}$$

the equation of the unit sphere. Thus, with these variables the state of the system is specified by a point on the unit sphere, the azimuth of which determines the relative phase of the dipole moment with respect to the applied field, and the colatitude of which determines the energy. The upper and lower states are mapped onto the north and south poles, respectively. Conversely, each point on the sphere determines a particular linear combination of states (3.3) to within a phase factor which has no meaning. Thus, there is a 1:1 correspondence between points on the sphere and physical states of the atom.

If we define a resonance parameter

$$\alpha \equiv \Omega - \omega, \tag{3.14}$$

then substitution of the ansatz (3.10) into the Eqs. (3.7) results in

$$\dot{x} = \alpha y + (2\mu/\hbar) z(t) E(t) \sin\Omega t\,, \tag{3.15a}$$

$$\dot{y} = -\alpha x + (2\mu/\hbar) z(t) E(t) \cos\Omega t\,, \tag{3.15b}$$

$$\dot{z} = -(2\mu/\hbar)(x \sin\Omega t + y \cos\Omega t) E(t). \tag{3.15c}$$

This Cartesian form is convenient for use in obtaining explicit analytic solutions, but for some purposes it is convenient to express these equations in terms of the azimuth angle ϕ and the colatitude angle θ on the unit sphere reducing the system (3.15) to two coupled equations

$$\dot{\theta} = (2\mu/\hbar) \sin[(\omega + \alpha)t + \phi(t)] E(t), \tag{3.16a}$$

$$\dot{\phi} = -\alpha + (2\mu/\hbar) \cot\theta \cos[(\omega + \alpha)t + \phi(t)] E(t). \tag{3.16b}$$

Thus far, our results have been exactly equivalent to the usual semiclassical radiation theory except that we have not resorted to time-dependent perturbation theory. However, the present theory differs from conventional treatments in that here we will include radiation reaction in the driving field, i.e., we will take

$$E(t) = E_{\text{ext}}(t) + E_{\text{RR}}(t), \tag{3.17}$$

where $E_{\text{ext}}(t)$ is the external applied field, and $E_{\text{RR}}(t)$ is the radiation reaction field.

Many discussions of radiation reaction exist,[16–18] but none of the discussions seems quite appropriate for the present case, though all arrive at the same final results. We have included in Appendix B a derivation appropriate to our present problem. The results of that calculation, in the dipole approximation, are that the effect of radiation reaction on an oscillating dipole is equivalent to a field

$$E_{\text{RR}}(t) = \frac{2}{3c^3}\frac{d^3}{dt^3} M(t) - \frac{4K}{3\pi c^3}\frac{d^2}{dt^2} M(t) \tag{3.18}$$

acting back on the dipole. This expression gives the contribution for all frequencies up to the frequency K. Above that frequency the fields will not be effective since they correspond to wavelengths smaller than the radius of the charge distribution. For a charge distribution with dimensions of the order of the Bohr radius a_0, the cutoff is given approximately by

$$K \simeq c/a_0 = 6 \times 10^{18}/\text{sec}. \tag{3.19}$$

We will discuss this point further in Sec. 5, and show how K can be calculated from the current distribution $\vec{J}(r, t)$ within the atom.

Since the dipole moment to a good approximation oscillates sinusoidally at the natural frequency, Eq. (3.18) can be approximated by

$$E_{\text{RR}}(t) \simeq -\frac{2\omega^2}{3c^3}\dot{M}(t) + \frac{4\omega^2 K}{3\pi c^3} M(t). \tag{3.20}$$

Then, if we assume that the external field is given by

$$E_{\text{ext}}(t) = E_0 \cos\Omega t, \tag{3.21}$$

the total field driving the dipole is

$$E(t) = E_0 \cos\Omega t - \frac{2\omega^2}{3c^3}\dot{M}(t) + \frac{4\omega^2 K}{3\pi c^3} M(t). \tag{3.22}$$

Substituting this back into the differential equations (3.15) and using the ansatz (3.10), we have

$$\dot{x} = \alpha y + (4\mu^2\omega^3/3\hbar c^3)(x \sin^2\Omega t + y \sin\Omega t \cos\Omega t) z$$

$$+ (8\omega^3 K \mu^2/3\pi\hbar c^3)(x \sin\Omega t \cos\Omega t - y \sin^2\Omega t) z$$

$$+(2\mu/\hbar)zE_0\sin\Omega t\cos\Omega t, \tag{3.23a}$$

$$\dot{y}=-\alpha x+(4\mu^2\omega^3/3\hbar c^3)(x\sin\Omega t\cos\Omega t+y\cos^2\Omega t)z$$
$$+(8\omega^3K\mu^2/3\pi\hbar c^3)(x\cos^2\Omega t-y\sin\Omega t\cos\Omega t)z$$
$$+(2\mu/\hbar)E_0\cos^2\Omega t\,z, \tag{3.23b}$$

$$\dot{z}=-(4\mu^2\omega^3/3\hbar c^3)(x^2\sin^2\Omega t+y^2\cos^2\Omega t$$
$$+2xy\sin\Omega t\cos\Omega t)-(8\mu^2K\omega^3/3\pi\hbar c^3)(x^2-y^2)$$
$$\times\sin\Omega t\cos\Omega t-(2\mu/\hbar)E_0(x\sin\Omega t\cos\Omega t$$
$$+y\cos^2\Omega t)-(8\mu^2K\omega^3/3\pi\hbar c^3)xy\cos 2\Omega t. \tag{3.23c}$$

These equations contain two types of terms, those which vary slowly with time, and those which oscillate rapidly at frequency 2Ω. These rapidly oscillating terms are often encountered in radiation theory; they are known in magnetic resonance theory as the "counter-rotating" terms and were discussed by Bloch and Siegert.[19] There terms oscillate so rapidly that their effect averages to zero in a half-cycle of the dipole's oscillation. We will then neglect those terms by averaging the equations over a time π/Ω. The resulting equations will be called the secular equations and will describe the slow change of the energy and of the phase of the dipole moment. The secular equations are

$$\dot{x}=\alpha y+\beta xz-\gamma yz, \tag{3.24a}$$

$$\dot{y}=-\alpha x+\beta yz+\gamma xz+\lambda z, \tag{3.24b}$$

$$\dot{z}=\beta(z^2-1)-\lambda y, \tag{3.24c}$$

where $$\lambda\equiv\mu E_0/\hbar, \tag{3.25}$$

$$\beta\equiv 2\mu^2\omega^3/3\hbar c^3, \tag{3.26}$$

$$\gamma\equiv 4K\omega^2\mu^2/3\pi\hbar c^3 \tag{3.27}$$

The parameter β is exactly one-half the Einstein A coefficient; it will appear presently that γ is related to the Lamb shift. As a check on the accuracy of this approximation, note that the first integral (3.13) is still an exact first integral of the secular equations (3.24).

The corresponding secular equations for the azimuth and the colatitude are

$$\dot{\phi}=-\alpha+\lambda\cos\phi\cot\theta+\gamma\cos\theta, \tag{3.28}$$

$$\dot{\theta}=\beta\sin\theta+\lambda\sin\phi.$$

This formalism differs from the usual semiclassical theory in taking the dipole moment of the atom to be an actual oscillating charge distribution which radiates according to the classical Maxwell equations. This radiation field then acts back on the atomic dipole moment to cause radiation reaction. The radiation given off by the atom undergoing spontaneous or stimulated emission will be proportional to $\dot{M}$, as in classical electromagnetic theory.

The predictions of this theory will not always be in exact quantitative agreement with QED, as we will see in Sec. 4; however, the area of agreement does seem to be remarkably wide. We will see that the two theories seem to be in complete agreement in their description of all experiments which have been carried out heretofore, and that they differ only in finer details, yet to be tested experimentally. We will be particularly interested in those cases where the results differ from the predictions of QED. These results will, at the very least, point out areas in which this method breaks down as an approximation technique for QED. The method of calculation has a certain plausibility of its own however, quite separate from QED, and may be regarded not merely as an approximation scheme to QED, but as a physical theory in its own right; one which we suggest as a possible alternative to QED in which the divergence difficulties are conspicuously missing, yet one which agrees with existing experiments. The areas in which their predictions differ thus correspond to new experiments capable of deciding between them.

Of course, one does not expect that this alternative theory can be correct in every respect; however, we believe it to be useful for suggesting new experiments. If new experiments show that this alternative theory contains just one "element of truth" that is not contained in present QED, then we would have the essential clue telling us in what direction QED must be modified.

4. SOLUTIONS OF SECULAR EQUATIONS

Solutions of the secular equations contain a great amount of detail. In order to show clearly the nature of these solutions we treat the various special cases first, gradually generalizing our analysis. By use of this approach we can see the ways in which the various parameters influence the solutions.

A. Strong Fields

The simplest case for which we can obtain nontrivial solutions is the strong-field case, i.e., the case in which the external applied field is near resonance and is so strong that radiation reaction is negligible. This corresponds to neglecting α, β, and γ when compared with λ. The secular equations are then

$$\dot{x} = 0,$$
$$\dot{y} = \lambda z, \tag{4.1}$$
$$\dot{z} = -\lambda y.$$

Assuming that the atom was in its ground state at $t = 0$, the solutions are

$$x(t) = 0,$$
$$y(t) = -\sin\lambda t, \tag{4.2}$$
$$z(t) = -\cos\lambda t.$$

Physically this says that the atom alternately absorbs and then reemits radiation at a rate which is directly proportional to the driving field. This sort of behavior has been observed in microwave spectroscopy where spontaneous emission is negligible, and is derivable from conventional radiation theory.

B. Pure Spontaneous Emission

The second case which we want to look at in detail is pure spontaneous emission when the Lamb shift is negligible. In general, the Lamb shift may be large compared to the natural line width, but in order to better understand the qualitative effects of spontaneous emission we will ignore the Lamb shift for the present.

The secular equations in this case are

$$\dot{x} = \beta x z,$$
$$\dot{y} = \beta y z, \tag{4.3}$$
$$\dot{z} = \beta(z^2 - 1).$$

The $\dot{z}$ equation can be integrated immediately and we get the solutions,

$$x(t) = \cos\phi_0 \,\mathrm{sech}\beta(t - t_0),$$
$$y(t) = \sin\phi_0 \,\mathrm{sech}\beta(t - t_0), \tag{4.4}$$
$$z(t) = -\tanh\beta(t - t_0),$$

This is spontaneous emission, but in a rather different form from that found in QED (see Fig. 1). Note that if the atom is exactly in the excited state, i.e., $z = 1$, then $\dot{x} = \dot{y} = \dot{z} = 0$. The atom will remain forever in the excited state! This is rather disconcerting until we realize that this is a point of metastable equilibrium, so that even the very slightest perturbation would be sufficient to cause the system to decay as given by (4.4). In any experiment such perturbations will, of course, be unavoidable. This differs from QED, which predicts that spontaneous decay will begin immediately. If isolated atoms could be pumped very close to the excited state, an experimentally observable delay should occur before spontaneous emission begins, if this theory is correct. This has not been confirmed or disproved by existing experiments. We will discuss this further in Sec. 6.

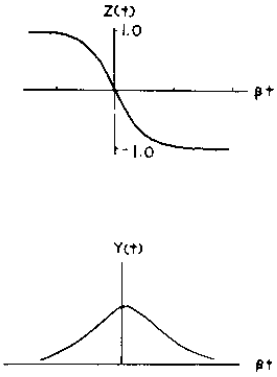


FIG. 1. The time dependence of the energy and of the amplitude of the dipole moment for spontaneous emission.

This theory predicts the proper decay constant, the Einstein A coefficient. For long-term decay,

$$W(t) = -\tfrac{1}{2}\hbar\omega \tanh\beta t$$
$$\to -\tfrac{1}{2}\hbar\omega(1 - 2e^{-At}), \quad t \to \infty \tag{4.5}$$

where $A \equiv 2\beta$. This is exactly the QED result.

These differences between the present theory and QED are observable only when we can pump the atom near the upper state. If the atom is excited only slightly above the ground state, these results are identical to those found in the conventional theory.

Another comparison can be made with the line shape predicted by the two theories. The line shape is given by $|M(\omega)|^2$, and

$$M(\omega) \equiv \frac{\mu}{2\pi}\int_{-\infty}^{0} e^{i\omega t}\,\mathrm{sech}\beta t\,\cos\omega_0 t\,dt. \tag{4.6}$$

If $\omega \gg \beta$, then

$$|M(\omega)|^2 \simeq (\pi\mu^2/8\beta^2)[\mathrm{sech}^2(\pi/2\beta)(\omega + \omega_0) + \mathrm{sech}^2(\pi/2\beta)(\omega - \omega_0)], \tag{4.7}$$

so the line shape is given by a hyperbolic secant squared rather than a Lorentzian. These line-shape differences might be distinguished by a visibility-curve measurement such as those performed by Michelson,[20] if we could eliminate the Doppler broadening and pump the atoms close enough to the excited state so that the entire decay would be present in the radiation.[21]

Having seen this much concerning the phenomena of spontaneous emission in the formalism, we will

go on to investigate the effects of the other terms in our equations.

C. Spontaneous Emission with the Lamb Shift

We will now solve the general secular equations with no applied field:

$$\dot{x} = \beta xz - \gamma yz,$$

$$\dot{y} = \beta yz + \gamma xz, \tag{4.8}$$

$$\dot{z} = \beta(z^2 - 1).$$

The $\dot{z}$ equation is identical with that for spontaneous emission with no Lamb shift, so we can immediately write down the result,

$$z(t) = -\tanh\beta(t - t_0). \tag{4.9}$$

Rather than solving Eqs. (4.8) for $x(t)$ and $y(t)$, it is better to return to the angular form of the secular equations (3.28). Equation (3.28) becomes, in the present case,

$$\dot{\phi}(t) = \gamma\cos\theta$$

$$= \gamma z(t)$$

$$= -\gamma\tanh\beta(t - t_0). \tag{4.10}$$

But since

$$|\dot{\phi}| \ll \omega|\phi| \tag{4.11}$$

and $$\dot{M} + i\omega M = i\mu\omega\sin\theta\, e^{i[\omega t + \phi(t)]}, \tag{4.12}$$

we see that $\dot{\phi}(t)$ measures a frequency shift in the oscillations of the dipole moment, and of the emitted radiation. When the atom is near the upper state, the dipole is oscillating at frequency $\omega + \gamma$, while it oscillates at frequency $\omega - \gamma$ near the lower state, thus the emitted radiation is frequency modulated! This result again differs with QED, where the frequency of the emitted radiation is constant.

This result is consistent with the observation that the Lamb shift seems to raise the S levels, however. In experiments in which the transition observed has the S level as the lower of the two levels involved, it is observed that the transition frequency is lowered. Our theory gives that frequency as $\omega - \gamma$ near the lower level, which is all that will be observed in an experiment in which incoherent radiation is absorbed. In experiments in which the S level is the upper level, the transition frequency is increased, which again agrees with our result, since we get $\omega + \gamma$ in that case. At this point it is not obvious why we should pick out the frequency near the S level in the second case, but it is due to many level effects which we will discuss in Sec. 5.

For comparison with QED, we can again calculate the line shape. Integrating (4.10), we find

$$\phi(t) = (\gamma/\beta)\ln\operatorname{sech}\beta(t - t_0), \tag{4.13}$$

which we can substitute into Eq. (3.10) to determine $M(t)$:

$$M(t) = \mu\,\mathrm{Re}[(\operatorname{sech}\beta t)^{1 + i\gamma/\beta} e^{i\omega_0 t}]. \tag{4.14}$$

The Fourier transform is then obtained by use of the change of variables

$$\tanh\beta t \equiv 1 - 2u. \tag{4.15}$$

Keeping only the positive frequency part, we have for the line shape

$$|M(\omega)|^2 \simeq \frac{\mu^2\pi}{\beta\gamma}\sinh\frac{\gamma}{\beta}\left[\cosh\frac{\gamma}{\beta} + \cosh\frac{(\omega - \omega_0)}{\beta}\right]^{-1}. \tag{4.16}$$

This line shape is illustrated in Fig. 2. Again we might be able to test this result with a Michelson visibility experiment if we could guarantee that the excitation was such that we got the entire transition.

D. Applied Field, No Lamb Shift

We will now investigate, in detail, the solutions when there is a monochromatic applied field. We will first neglect the frequency shift. In this case the secular equations are

$$\dot{x} = \beta xz + \alpha y,$$

$$\dot{y} = \beta yz + \lambda z - \alpha x, \tag{4.17}$$

$$\dot{z} = \beta(z^2 - 1) - \lambda y.$$

These equations are linearized by the following ansatz:

$$x \equiv f/h, \quad y \equiv g/h, \quad z \equiv -\dot{h}/\beta h. \tag{4.18}$$

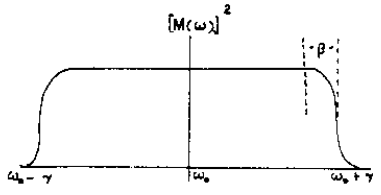


FIG. 2. The line shape for spontaneous emission including the time-dependent frequency shift.

With this substitution, Eqs. (4.17) reduce to

$$\dot{f} = \alpha g,$$

$$\dot{g} = -(\lambda h/\beta) - \alpha f, \tag{4.19}$$

$$\ddot{h} = \beta^2 h + \beta\lambda g.$$

These are linear equations which are easily solved though the solutions for $x(t)$, $y(t)$, and $z(t)$ are quite complicated. For example, the solution for $z(t)$ when $z(0) = 1$ is

$$z(t) = N(t)/D(t),$$

where

$$N(t) = -(a/\beta)[\beta(a^2 + \lambda^2 - \beta^2)\cos bt + b(a^2 - \beta^2) \times \sin bt + \beta(\lambda^2 - \beta^2 - b^2)\cosh at + a(b^2 + \beta^2)\sinh at] \tag{4.20}$$

and $$D(t) = [a(a^2 - \beta^2)\cos bt - (\beta a/b)(a^2 + \lambda^2 - \beta^2)\sin bt + a(b^2 + \beta^2)\cosh at - \beta(b^2 + \beta^2 - \lambda^2)\sinh at].$$

The oscillation frequency b is

$$b \equiv \{-\tfrac{1}{2}(\beta^2 - \lambda^2 - \alpha^2) + \tfrac{1}{2}[(\alpha^2 + \lambda^2 - \beta^2)^2 + 4\alpha^2\beta^2]^{1/2}\}^{1/2}, \tag{4.21}$$

while the damping constant a is

$$a \equiv \{\tfrac{1}{2}(\beta^2 - \lambda^2 - \alpha^2) + \tfrac{1}{2}[(\alpha^2 + \lambda^2 - \beta^2)^2 + 4\alpha^2\beta^2]^{1/2}\}^{1/2}. \tag{4.22}$$

The solutions for $x(t)$ and $y(t)$ are of the same form. While it is impossible to understand the details of this solution by examination, we can say at least that it is a damped nonsinusoidal oscillation with oscillation frequency b and damping constant a. In all cases for which $a \neq 0$ the solution will decay exponentially to a steady-state asymptotic value. In fact, we can see

$$\lim z(t) = -a/\beta, \quad \text{as } t \to \infty \ . \tag{4.23}$$

To obtain a better understanding of these rather complex analytic solutions, we will further analyze the differential equations themselves and study some graphical solutions obtained by use of an analog computer.

Consider the dependence of the oscillation frequency b, and the decay constants, on the parameters α and λ. The spontaneous emission parameter is a constant for a given transition. Expanding (4.22), we can write it in the form

$$\lambda^2/(\beta^2 - a^2) - \alpha^2/a^2 = 1 \ . \tag{4.24}$$

This is an hyperbola and has solutions only for

$$a \leq \beta \ , \tag{4.25}$$

thus the transients always die out more slowly than spontaneous emission. We can treat (4.21) in the same way. The oscillation frequency b then satisfies

$$\lambda^2/b^2 + \beta^2 + \alpha^2/b^2 = 1, \tag{4.26}$$

the equation of an ellipse. These curves are illustrated in Fig. 3. Using this graph we can read off the oscillation frequency and the damping constant for any set of the parameters α and λ.

Note that we have qualitatively different solutions for the case $\alpha = 0$, exact resonance. At resonance with a field strength satisfying $\lambda < \beta$, the oscillation frequency is zero. Above the "critical field," $\lambda > \beta$, there is no damping; the solutions oscillate forever. This is true only for exact resonance, however. Any slight value of α will cause the system to decay slowly to a steady-state value.

We can also use the differential equations to determine the asymptotic steady-state solutions. We obtain these solutions by requiring

$$\dot{x} = 0 = \beta x z + \alpha y \ ,$$

$$\dot{y} = 0 = \beta y z + \lambda z - \alpha x \ , \tag{4.27}$$

$$\dot{z} = 0 = \beta(z^2 - 1) - \lambda y \ .$$

There are two sets of solutions, one set in the northern hemisphere and another in the southern hemisphere. We have already seen in (4.23) that the asymptotic solution is $z = -a/\beta$, so that the asymptotic solutions are

$$x = -(\alpha/\lambda)[(\beta/a) - (a/\beta)],$$

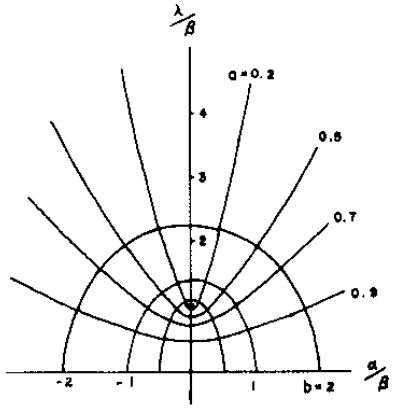


FIG. 3. The decay constants and oscillation frequencies for transients after sudden turn on of the driving field.

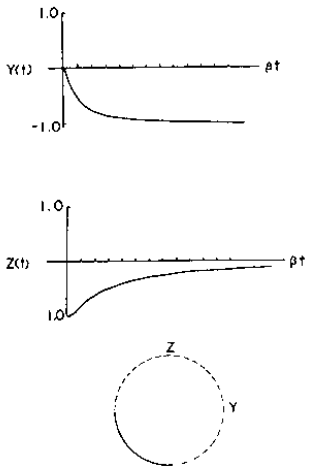


FIG. 4. Solutions of the secular equations for $\alpha=\gamma=0$ and $\lambda=0.99\beta$.

$$y=-(\beta/\lambda)(1-\alpha^2/\beta^2), \tag{4.28}$$

$$z=-\alpha/\beta .$$

The point in the northern hemisphere is a point of unstable equilibrium.

Figures 4–6 show graphs of solutions for $x(t)$, $y(t)$, and $z(t)$ for various values of the parameters α and λ. These graphs were obtained by solving the differential equations (4.17) directly with an analog computer.

E. Strong Field with Lamb Shift

We have already seen that the effect of the Lamb shift is to cause the frequency of the atomic transition to be raised when the atom is near the upper state, and lowered when the atom is near the lower state. We would expect such a frequency shift to cause effects which are evident with fields so strong that spontaneous emission can be neglected.

In this case the secular equations become

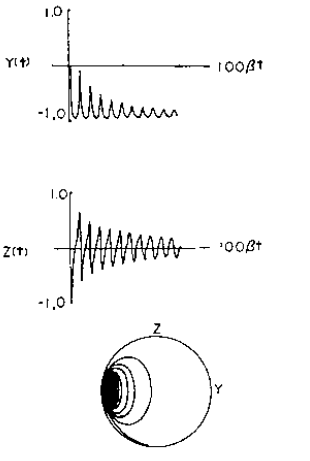


FIG. 5. Solutions of the secular equations for $\alpha<0.01\beta$, $\lambda\simeq1.01\beta$, and $\gamma=0$.

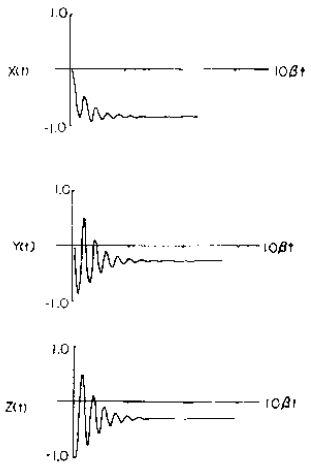


FIG. 6. Solutions of the secular equations for strong applied field off resonance, $\gamma=0$, $\alpha=\beta$, and $\lambda=3\beta$.

$$\begin{aligned}\dot{x}&=\gamma yz\ ,\\ \dot{y}&=\gamma xz+\lambda z\ ,\\ \dot{z}&=-\lambda y\ ,\end{aligned} \tag{4.29}$$

where we have assumed the applied field to be resonant with the unshifted transition frequency.

There is an immediate first integral by eliminating z between the $\dot{x}$ and $\dot{y}$ equations. If we assume $x(0)=y(0)=0$, the integral is

$$(x+\lambda/\gamma)^2+y^2=(\lambda/\gamma)^2\ . \tag{4.30}$$

This says that the projection of the system point orbit in the xy plane is a circle with its center at $x=-\lambda/\gamma$, $y=0$ and radius of λ/γ. This orbit lies entirely on the sphere only if $\lambda<\frac{1}{2}\gamma$. By use of the equation of the sphere (3.13), we can determine the other two projections of the orbit. The xz projection is an hyperbola, but the yz projection is a quartic, an ellipse in terms of y and z^2. These projections are shown in Fig. 7 for vari-

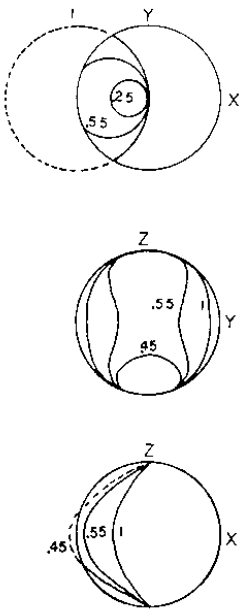


FIG. 7. The system-point orbits for $\alpha=\beta=0$, various λ/γ.

ous values of the applied field strength.

We can actually obtain an analytic solution for $x(t)$, $y(t)$, and $z(t)$ in this case. Combining the first integral with the $\dot{z}$ equation (4.29), we have an equation in the form for solution in terms of elliptic functions. From the tables[22] we have

$$z(t) = -cn(\lambda t \,|\, \gamma/2\lambda), \quad \gamma > \tfrac{1}{2}\gamma \ . \tag{4.31}$$

By use of the Eqs. (4.29) and the identities for elliptic functions, we can determine the corresponding solutions for $x(t)$ and $y(t)$:

$$x(t) = -(\gamma/2\lambda)sn^2(\lambda t \,|\, \gamma/2\lambda) \ , \tag{4.32}$$

$$y(t) = sn(\lambda t \,|\, \gamma/2\lambda)dn(\lambda t \,|\, \gamma/2\lambda) \ . \tag{4.33}$$

In the region $\lambda < \frac{1}{2}\gamma$ the solutions are

$$x(t) = -(2\lambda/\gamma)sn^2(\tfrac{1}{2}\gamma t \,|\, 2\lambda/\gamma) \ , \tag{4.34}$$

$$y(t) = (2\lambda/\gamma)sn(\tfrac{1}{2}\gamma t \,|\, 2\lambda/\gamma)cn(\tfrac{1}{2}\gamma t \,|\, 2\lambda/\gamma) \ , \tag{4.35}$$

$$z(t) = -dn(\tfrac{1}{2}\gamma t \,|\, 2\lambda/\gamma) \ . \tag{4.36}$$

A typical case is illustrated in Fig. 8.

In the limit of a strong field, i.e., $\lambda \gg \gamma$, the solutions reduce to those given in (4.2) as expected, i.e.,

$$\lim z(t) = cn(\lambda t \,|\, 0) = -\cos\lambda t, \quad \text{as } \lambda \to \infty \tag{4.37}$$

The solution is also simple in the case $\lambda = \frac{1}{2}\gamma$ when we have

$$z(t) = -cn(\tfrac{1}{2}\gamma t \,|\, 1) = -\operatorname{sech}\tfrac{1}{2}\lambda t, \quad \lambda = \tfrac{1}{2}\lambda \ . \tag{4.38}$$

This last case is the only nonoscillatory solution to Eqs. (4.29); we must in general include spontaneous emission before we get damping or steady-state solutions.

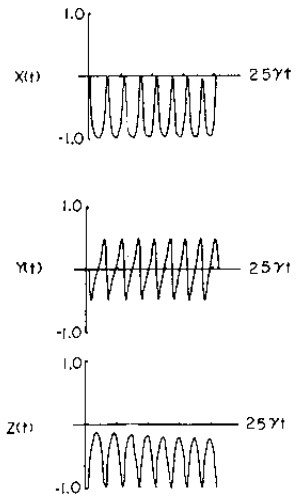


FIG. 8. Solutions of the secular equations for $\alpha = \beta = 0$, and $\lambda < \frac{1}{2}\gamma$.

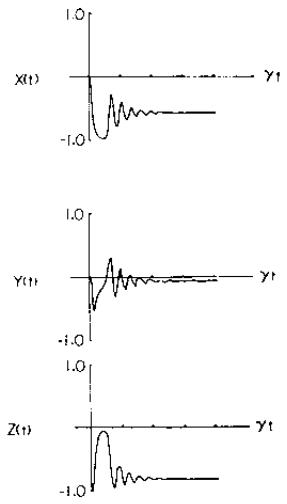


FIG. 9. Solutions of the general secular equations at resonance with a weak applied field, $\alpha = 0$, $\beta = 0.1\gamma$, and $\lambda \simeq \frac{1}{2}\gamma$.

We can obtain analytic solutions also for the case in which the applied field is off-resonance. In this case the secular equations are

$$\begin{aligned} \dot{x} &= -\gamma yz + \alpha y \ , \\ \dot{y} &= \gamma xz + \lambda z - \alpha x \ , \\ \dot{z} &= -\lambda y \ . \end{aligned} \tag{4.39}$$

Again there is a first integral which gives a projection of the system point orbit

$$2\lambda x/\gamma + (1 + \alpha/\gamma)^2 = (z - \alpha/\gamma)^2 \ ; \tag{4.40}$$

the other two projections can be obtained as before. These orbits have been graphed for various values of α, and $\lambda = \frac{1}{2}\gamma$.[23] There is an asymmetry between positive and negative values of α. This is because a field with its frequency such that $\alpha = -\gamma$ will be at resonance with the shifted frequency when the atom is in its ground state, while a field with its frequency such that $\alpha = \gamma$ will be at resonance when the atom is in the excited state.

Substituting Eq. (4.31) into the $\dot{z}$ equation (4.30), we get a differential equation which is again in the form for solution by elliptic functions. The solution is too complicated to be of much value, so we will rely on the analog solutions, even though we could obtain an exact analytic solution.

F. General Solutions

We have thus far been unable to obtain analytic solutions to the secular equation in the most general case. By use of the analog computer we have been able to obtain graphical solutions, however; those are shown in Figs. 9–11.

For the case of exact resonance, $\alpha = 0$, we are able to reduce the secular equations to quadrature, though we are unable to perform the resulting in-

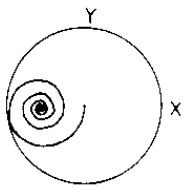
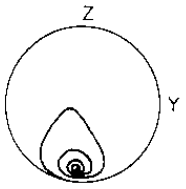


FIG. 10. The system-point orbits for $\alpha = 0$, $\beta = 0.1\gamma$, and $\lambda \simeq \frac{1}{2}\gamma$.



tegral.[24]

There are several things we can learn about these general solutions. The analog plots seem to indicate that for $\alpha = 0$, the critical field is $\lambda_c \simeq \frac{1}{2}\gamma$. Above this field, the solutions are oscillatory and there seems to be no damping. Below this field, the solutions are damped to their steady-state values in times long compared with $(\beta)^{-1}$. We can determine the steady-state values. At resonance the condition for steady-state solutions is

$$\begin{aligned} 0 &= \dot{x} = \beta xz - \gamma yz \ , \\ 0 &= \dot{y} = \beta yz + \gamma xz + \lambda z \ , \\ 0 &= \dot{z} = \beta(z^2 - 1) - \lambda y \ . \end{aligned} \tag{4.41}$$

The solutions are

$$\begin{aligned} x_\infty &= -\gamma\lambda/(\gamma^2 + \beta^2) \ , \\ y_\infty &= -\beta\lambda/(\gamma^2 + \beta^2) \ , \\ z_\infty &= -[1 - \lambda^2/(\gamma^2 + \beta^2)]^{1/2} \ . \end{aligned} \tag{4.42}$$

5. GENERAL FORMALISM

The formalism which we have developed and applied in Secs. 3 and 4 uses an interpretation of quantum mechanics that differs from the Copenhagen interpretation which is normally used. When the dipole moment of the atom is taken to be the actual charge distribution which gives off the atomic radiation, we are using the "Schrödinger interpretation." Schrödinger proposed in 1926 that the quantities

$$\rho = e\,|\Psi(r,t)|^2 \tag{5.1a}$$

and

$$\vec{J} = (e/m)\,\mathrm{Re}(\Psi^* \vec{P} \Psi), \tag{5.1b}$$

where e is the electronic charge, Ψ is the atomic wave function, and P is the momentum operator, be interpreted as the actual electric charge density and current density associated with the atom.[2] By about 1930, this interpretation was superseded by the present interpretation. The reasons for dropping Schrödinger's interpretation are not altogether clear, though some are known.

One objection raised against this interpretation is that in the case of more than one particle, the charge distribution, as defined in Eqs. (5.1), is not defined in ordinary 3-dimensional space for an N-electron system, since the wave function for such a system is a function of the $3N$ components of the coordinates of the N particles. The charge distribution defined in (5.1) is only for a one-electron system, however, and is not generalized to a N-electron system by simply replacing $\Psi(r,t)$ by $\Psi(r_1, r_2, \ldots, r_N, t)$. Instead the charge density of the N-particle system is given as the sum of the charge densities of the individual particles. The charge density due to electron number 1 in such a system would be

$$\rho_1(r,t) = e \int |\Psi(r, r_2, r_3, \ldots, r_N)|^2 \times d^3r_2 d^3r_3 \cdots d^3r_N; \tag{5.2a}$$

for electron 2,

$$\rho_2(r,t) = e \int |\Psi(r_1, r, r_3, \ldots, r_N, t|^2 d^3r_1 \times d^3r_3 d^3r_4 \cdots d^3r_N, \tag{5.2b}$$

etc.; and the total charge density for the N-electron system would be

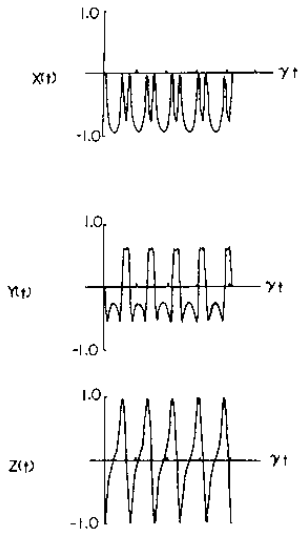


FIG. 11. Solutions of the general secular equations with a strong applied field, $\alpha = 0$, $\beta = 0.1\gamma$, and $\lambda > \frac{1}{2}\gamma$.

$$\rho(r,t)=\sum_{l=0}^{N}\rho_l(r,t)\,. \tag{5.2c}$$

Likewise, the contribution to the current from the first electron would be

$$\vec{J}_1(r,t)=(e/m)\mathrm{Re}\int\Psi^*(r,r_2,r_3,\ldots,r_N,t)\times\vec{P}\Psi(r,r_2,r_3,\ldots,r_N,t)d^3r_2d^3r_3\cdots d^3r_N, \tag{5.3a}$$

and similarly for all of the other electrons so that the total current would be

$$\vec{J}(r,t)=\sum_{l=0}^{N}J_l(r,t). \tag{5.3b}$$

These quantities represent a generalization of the quantities defined in (5.1) which are defined in ordinary 3-dimensional space. It is also easily shown that as a result of the N-particle Schrödinger equation they satisfy the usual continuity relation,

$$\nabla\cdot\vec{J}+\frac{1}{c}\frac{\partial\rho}{\partial t}=0. \tag{5.4}$$

Another argument often heard is that this and all other semiclassical theories are incapable of predicting spontaneous emission and the Lamb shift. We have seen in Secs. 3 and 4 that such is not the case; we have in fact described just those phenomena as well as other distinctly "quantum" phenomena.

There is one argument which is not entirely answered at this point. The usual quantum-mechanical solution for the free-electron problem predicts a spreading wave packet, thus according to the interpretation (5.1), we would have a solution for the free electron which is not stable, but spreads indefinitely over all space. This is not really a correct description of an electron according to this formalism, however. An electron is charged and thus has an associated field which reacts back on the electron. We must solve the problem of the electron interacting with its own field. It is possible for such a charge distribution to be bound and nonradiating. Bohm and Weinstein[25] have described such charge distributions and have shown that in fact the condition that they be bound is exactly the condition that they be nonradiating. Clearly, any description of the free electron must necessarily include spin and thus use the Dirac equation for the electron. This causes no essential difficulty since the Eqs. (5.1) are easily generalized for the Dirac equation. The resulting equations remain to be solved, but the essential point is that this interpretation cannot be rejected due to this argument until this problem is solved.

The secular equations (3.24) were derived using this Schrödinger interpretation, but only in the dipole approximation, for linearly polarized applied fields, and a "two-level atom." These restrictions and approximations are not necessary but are made only to simplify the analysis so that analytic solutions can be obtained. More complicated cases have been treated. The treatment of applied fields with general polarization is straightforward and has been carried out.[24] The result is that the energy and magnitude of the dipole moment have the same time dependence as in the linearly polarized case. The problem has also been treated without the dipole approximation, actually using (5.1) to describe the oscillating charge distribution.[26] The result is that the equations reduce to exactly those we derived in Sec. 3, except that a definite numerical value is obtained for the constant γ of Eq. (3.27), the Lamb shift. This constant which diverges in the point dipole approximation converges nicely for the charge distribution (5.1), which is spread over the entire atom. The numerical value of this constant has been compared with the Lamb-shift experiments and found to agree within experimental error.[26] Finally, the extension to n levels has been carried out.[25] The equations in that case are somewhat more complex, but can be seen to yield the familiar cascade of "photons" as an electron descends from a higher excited state through successive lower states to the ground state. When only a few levels are involved in the transition, the problem is easily solved with an analog computer, as we did in Sec. 4.

One other case in which we can easily obtain solutions is the case in which the applied field is not cw but pulsed or amplitude modulated. If the amplitude of the applied field varies slowly compared with the spontaneous emission time $(\beta)^{-1}$, then the atom will remain in its steady state [(4.43) or (4.28)]. The steady state slowly changes according to the value of $\lambda=\lambda(t)$:

$$\lambda(t)=\mu E_0(t)/\hbar, \tag{5.5}$$

where $E_0(t)$ is the slowly varying amplitude of the applied field. If the field is very strong, i.e., $\lambda\gg\beta$ and $\lambda\gg\gamma$, then yet another solution can be obtained for an important special case. If the field is of the form

$$E_{\mathrm{ext}}(t)=E_0\ \mathrm{sech}(E_0t/h)\ \cos\omega t, \tag{5.6}$$

then the secular equations can be written as

$$\dot{x}=0, \tag{5.7a}$$

$$\dot{y} = \lambda(t)z, \tag{5.7b}$$

$$\dot{z} = -\lambda(t)y, \tag{5.7c}$$

$$\lambda(t) = \lambda_0 \operatorname{sech} \lambda_0 t, \tag{5.7d}$$

that they received a π pulse of the radiation in traversing the illuminated region, then the atoms would emerge with the proper initial conditions so that their subsequent spontaneous decay pulse should have a line shape like that given in Fig. 2. The width of the line would be twice the Lamb shift of the given transition unless the Lamb shift happened to be small for the particular transition, in which case the line width would be just that given by the usual theory but the line shape would be a hyperbolic secant rather than the Lorentzian (see Fig. 1). There seems to be no reason in principle why the experiment could not be carried out this way, the main difficulty would be obtaining a suitable two-level atom and an accompanying coherent source which can deliver a π pulse.

An interesting point concerning this experiment is that QED does not predict exactly a Lorentzian line shape for a spontaneous decay; the Lorentzian line shape is a result of the Wigner-Weisskopf approximation. We have solved the problem of spontaneous decay using QED without time-dependent perturbation theory in order to investigate this problem.[24] The method used was similar to that given by Kroll.[28] We found that, indeed, spontaneous decay is not exponential for long times, but rather the amplitude of the upper state falls off as t^{-2} eventually. This correction term is extremely small, about one part in 10^9, so that for all practical purposes, the decay is over before the correction is appreciable compared with the exponentially decaying term. Thus, as far as any feasible experiment is concerned, QED predicts a Lorentzian line shape.

As we pointed out in Sec. 4, this π pulse is extremely important in any experiment which can hope to find these effects. If the atoms are excited by an ordinary incoherent source, they will remain near the ground state and only the exponential tail which agrees with QED will be observed. If we are able to prepare good enough π pulses, another type of experiment becomes feasible. The formalism predicts a metastability of the system in the excited state, thus there should be a delay before the decay takes place if we prepare the system sufficiently near the excited state. The necessary preparation is quite exacting, however, as we must get the atom in a state with $z(0) > 0.94$ in order to double the decay time, and with $z(0) > 0.9999$ in order to get five times the decay time.

The very detailed solutions presented in Sec. 4 offer all sorts of additional possibilities for experiments. Their description of the nonlinear interference which occurs when spontaneous and stimulated emission occur simultaneously is especially interesting. It suggests that we might

where $\lambda_0 \equiv \mu E_0/\hbar$. (5.7e)

These are easily solved to obtain

$$z(t) = \pm \tanh \lambda_0 t,$$

$$y(t) = \mp \lambda_0 \operatorname{sech} \lambda_0 t = \mp \lambda(t).$$

This is the hyperbolic secant which was seen experimentally by Hahn and McCall.[27] This is the one pulse shape which is preserved on absorption and reemission.

Other types of coherent excitation are easily described by analog computation.

6. COMPARISON WITH QED EXPERIMENTAL TESTS

We have described a method for doing calculations in quantum radiation theory. In Sec. 4 we have described rather complex phenomena in a mathematically simple and intuitively appealing way. In Sec. 5 we have shown that the method is quite general and capable of treating a wide range of problems. If we use the Dirac equation for calculating the current, we can treat relativistic problems by this formalism. A great many calculations, for example, the Klein-Nishina formula, are normally calculated semiclassically and thus have already been described by a variant of this formalism. Even such "far out" phenomena as light-light scattering can be described semiclassically using the Dirac current.

The method which we have described is not identical with QED. In fact, the two theories disagree in their predictions of some phenomena. These disagreements are, however, always in fine details which have not yet been tested by experiment. Some of these fine details would seem to be subject to experiment with existing technology. These experiments are interesting not only as a test of the semiclassical theory but as a test of QED. Even if this semiclassical formalism does not turn out to be *the* ultimate correct theory it might well point out an experiment in which QED fails. In that case we will have a vital clue as to how QED should be modified.

There are several possible types of experiments which are suggested by the calculations of Sec. 4. The most obvious seems to be the line shape which under certain experimental conditions can differ greatly from the Lorentzian predicted by the Wigner-Weisskopf theory. Let us go into a possible experiment of this type in a little detail. Of course, Doppler broadening and various types of homogeneous broadening in a solid make it dif-

ficult to do any sort of natural line-shape experiment in a gas or solid. These difficulties could be overcome by using an atomic beam. If a beam of our "two-level atoms" passed through a region of coherent illumination at the proper velocity so do some experiments observing the interference between an applied field and the stimulated field which it generates in resonant scattering. Definite asymptotic phase relations are predicted in Eq. (4.42) for this problem. The corresponding QED calculations do not seem to have been carried out at this point.

These few examples illustrate the sort of possibilities which exist. It is hoped that the detailed solutions of Sec. 4 will suggest other possibilities perhpas simpler and more direct than those suggested above.

ACKNOWLEDGMENTS

Part of this work was carried out while one of us was a National Science Foundation Graduate Fellow. The analog computations were carried out on the Washington University Computing facilities supported by NSF Grant G-222 96. Finally, we would like to thank Centre College, Danville, Ky., for the use of their facilities during part of the preparation of this manuscript for publication.

APPENDIX A: TYPICAL VALUES OF RADIATION PARAMETERS

The following table gives the values of the radiation parameters for two special cases: (i) the $1S-2P$ transition in hydrogen, and (ii) the sodium D lines. The values are theoretically derived in the case of hydrogen, but in the case of sodium we are unable to locate reliable theoretical estimates of these parameters, so that we must rely on experimental measurements of the lifetime of the P states to determine the spontaneous emission parameter and thus the dipole moment. The value for the Lamb shift is based on the known dependence of the Lamb shift on its parameters. This value of the Lamb shift is only meant as an order of magnitude estimate. The critical fields E_β and E_γ are the fields at which the stimulated processes are able to overcome spontaneous emission and the radiative shift, respectively. The P_γ is just the power in a plane wave with the electric field E_γ.

TABLE I. Typical values of the radiation parameters.

Parameter	Hydrogen $1S$-$2P$	Sodium $3P$-$3S$
$\nu=\omega/2\pi$	2.5×10^{15}/sec	5×10^{14}/sec
$\beta=\frac{1}{2}A$	3×10^{8}/sec	3×10^{7}/sec
μ	1.29 ea	2.46 ea
γ	8.7×10^{9}/sec	3×10^{8}/sec
$E_\beta=\hbar\beta/\mu$	29 V/cm	1.5 V/cm
$E_\gamma=\hbar\gamma/2\mu$	360 V/cm	8 V/cm
P_γ	170 W/cm^2	0.1 W/cm^2

APPENDIX B: RADIATION REACTION

The radiation problem in which we are interested is an oscillating dipole whose radiation field reacts back on the dipole changing its motion. We will solve this problem using Green's function. The results of this calculation are the same as those found elsewhere by other methods.[17]

It is convenient for this discussion to consider our system to be contained in a volume V with a closed surface S. We will define a set of normal field modes by the equations,[29]

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}_a)-k_a^2\vec{E}_a=0, \quad \text{in } V \tag{B1}$$

$$\text{and } \hat{n}\times\vec{E}_a=0, \quad \text{on } S, \tag{B2}$$

where $\hat{n}$ is the inward normal to S. The $\vec{E}_a(\vec{r})$ are normalized so that

$$\int\vec{E}_a\cdot\vec{E}_b\,dV=\delta_{ab}. \tag{B3}$$

A related set of functions $H_a(r)$ is defined by

$$\vec{\nabla}\times\vec{E}_a=k_a\vec{H}_a,$$

$$\vec{\nabla}\times\vec{H}_a=k_a\vec{E}_a. \tag{B4}$$

The electric and magnetic fields can be expanded in terms of these eigenfunctions,

$$\vec{E}(\vec{r},t)=-2\sqrt{\pi}\sum_a p_a(t)\vec{E}_a(r),$$

$$\vec{H}(\vec{r},t)=2\sqrt{\pi}\sum_a \omega_a q_a(t)\vec{H}_a(r). \tag{B5}$$

Substituting these expansions into Maxwell's equations, we find

$$p_a=\dot{q}_a,$$

$$\dot{p}_a+\Omega_a^2 q_a=J_a(t), \tag{B6}$$

$$\text{where } J_a(t)\equiv 2c\sqrt{\pi}\int d^3r\,\vec{J}(r,t)\cdot\vec{E}_a(r) \tag{B7}$$

$$\text{and} \qquad \Omega_a\equiv k_a c.$$

Combining these Eqs. (B.6), we have

$$\ddot{p}_a+\Omega_a^2 p_a=\dot{J}_a. \tag{B8}$$

Now if our current is due to an oscillating dipole

$$\vec{J}(r,t) = (1/c)\dot{\vec{P}}(r,t), \tag{B9}$$

where $P(r,t)$ is the polarization,

$$\dot{J}_a(t) = 2\sqrt{\pi}\int d^3r\ \dot{\vec{P}}(r,t)\cdot\vec{E}_a(r), \tag{B10}$$

which reduces, for a point dipole at the origin, to

$$\dot{J}_a(t) = 2\sqrt{\pi}\,\vec{E}_a(0)\cdot\ddot{M}(t). \tag{B11}$$

The amplitude $p_a(t)$ then satisfies

$$\ddot{p}_a + \Omega_a{}^2 p_a = 2\sqrt{\pi}\ \vec{E}_a(0)\cdot\ddot{\vec{M}}(t). \tag{B12}$$

The solution for the amplitude can be written down immediately:

$$p_a(t) = \frac{2\sqrt{\pi}}{\Omega_a}\int_{-\infty}^{t} dt'[\vec{E}_a(0)\cdot\ddot{\vec{M}}(t')]\sin\Omega_a(t-t'), \tag{B13}$$

where we have dropped the homogeneous solutions which correspond to external applied fields. The electric field is then given by

$$\begin{aligned}\vec{E}(r,t) &= -4\pi\sum_a \vec{E}_a(r)\int_{-\infty}^{t} dt'\frac{\sin\Omega a(t-t')}{\Omega_a} \\ &\quad\times\vec{E}_a(0)\cdot\ddot{\vec{M}}(t') \\ &\equiv \int_{-\infty}^{t} G(r,t;0,t')\ddot{\vec{M}}(t')\,dt'.\end{aligned} \tag{B14}$$

We need for our purposes only the component of the field which is parallel to the dipole, and only its value at the position of the dipole: Thus, we need calculate only

$$G_{xx}(0,t;0,t') = -4\pi\sum_a \frac{E_{ax}{}^2(0)}{\Omega_a}\sin\Omega_a(t-t'). \tag{B15}$$

In order to carry out this sum we go to the continuum limit by letting the cavity dimensions become very large. In addition, we impose a high-frequency cutoff which is necessary only because of our dipole approximation. Our results would be perfectly convergent if we used the finite charge distribution $\rho \equiv e\Psi^*\Psi$ rather than our dipole approximation. In fact, Crisp has carried out that calculation and found it to give results identical to ours, except that it determines the value of the cutoff parameter.[26]
In that case

$$\begin{aligned}G_{xx}(0,t;0,t') &= -\frac{4V}{c^3\pi}\int_0^K d\Omega\Omega\langle E_{ax}{}^2(0)\rangle_{\rm av}\sin\Omega(t-t'), \\ &= -\frac{4}{3\pi c^3}\int_0^K d\Omega\ \Omega\sin\Omega(t-t') \\ &= \frac{4}{3\pi c^3}\frac{\partial}{\partial t}\left(\frac{\sin K(t-t')}{(t-t')}\right),\end{aligned} \tag{B16, B18}$$

where K is the cutoff, which will be left unspecified.

In our Schrödinger interpretation, we expect the charge distribution to be spread over a volume with a radius a, the Bohr radius. Any radiation with wavelength shorter than this will not have appreciable effect on the charge distribution, so we must use a cutoff if we are to use a dipole approximation. The conventional Lamb shift derivation uses a cutoff at a wavelength of the Compton radius, and nothing that we do here will be affected by the value we choose for the cutoff.

The value of $\langle E_{ax}{}^2(0)\rangle_{\rm av}$, the average taken over the field modes, is

$$\langle E_{ax}{}^2(0)\rangle_{\rm av} = \tfrac{1}{3}\langle E_a{}^2(0)\rangle_{\rm av} = 1/3V. \tag{B17}$$

Thus, we have

which gives, for $E_x(0,t)$,

$$E_x(0,t) = \frac{4K}{3\pi c^3}\int_{-\infty}^{t} dt'\ddot{M}(t')\frac{\partial}{\partial t}\left(\frac{\sin K(t-t')}{(t-t')}\right). \tag{B19}$$

Writing the derivative in terms of t' we can carry out an integration by parts

$$\begin{aligned}E_x(0,t) &= -\frac{4K}{3\pi c^3}\ddot{M}(t) + \frac{4}{3\pi c^3}\int_{-\infty}^{t} dt' \\ &\quad\times\frac{\sin K(t-t')}{(t-t')}\frac{d^3}{dt'^3}M(t').\end{aligned} \tag{B20}$$

Now if we change to a new variable defined by $y = K(t-t')$ the integral becomes

$$\int_0^\infty dy\ \frac{d^3}{dt^3}M\left(t-\frac{y}{K}\right)\frac{\sin y}{y} \simeq \frac{\pi}{2}\frac{d^3}{dt^3}M(t), \text{ as } K\to\infty. \tag{B21}$$

Thus, the reaction field acting on the dipole is

$$E_x(0,t) = \frac{2}{3c^3}\frac{d^3}{dt^3}M(t) - \frac{4K}{3\pi c^3}\frac{d^2}{dt^2}M(t). \tag{B22}$$

This is exactly the expression we use in Eq. (3.17).

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