Presented at the Physics of Quantum Electronics course,
Telluride, Colorado, August 1977

ANCIENT HISTORY OF FREE ELECTRON DEVICES

by

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CONTENTS

BETATRONS AND SYNCHROTRONS	1
LINEAR ACCELERATORS	5
RADIATION FROM AN OSCILLATING CHARGE	8
BACK TO THE LABORATORY SYSTEM	12
THE UNDULATOR	18
THE CYCLOTRON-MASER EQUATIONS	19
SCHWARZ-HORA EFFECT (?)	28
REFERENCES	38

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We recall some early thinking about the production of light from high-energy electrons, that started with problems in the design of electron accelerators, in which the writer was an interested observer and (at Stanford, Berkeley, and Princeton in 1946-1960) a sometime participant. In fact, some of that work is being put on record here for the first time, since there was no encouragement to follow up such ideas then, beyond a few Seminar talks (although it has been inflicted since on generations of students, as pedagogical fodder and homework problems).

BETATRONS AND SYNCHROTRONS

Radiation of light due to acceleration of free electrons occurs naturally in Bremsstrahlung. In the early 1940's, when the betatron and synchrotron were being developed, it was realized that this same phenomenon would place an upper limit to the attainable energy from any such device other than a linear accelerator.

Iwanenko and Pomeranchuk (1944) gave a formula for the betatron based on the relativistic generalization of the Larmor power formula $P_L = (2e^2/3c^3)\dot{v}^2$, and Schwinger (1946, 1949) produced an immense calculation of radiation from betatrons and synchrotrons, giving its full spectral and directional distribution. The predicted radiation was of surprisingly high frequency, as confirmed experimentally by Elder, Langmuir, and Pollock (1948) on the G. E. synchrotron. For an electron in an orbit of radius R = 29.2 cm (orbital frequency 163.5 MHz) it reached the optical region with the appearance of a dull red spot at 30 Mev, progressing to brilliant bluish-white light at 80 Mev.

The reason why the emitted radiation is mostly in extremely high harmonics--millions of times the orbital frequency--can be seen as follows. A transversely accelerated electron with energy $E = \gamma mc^2$, $\gamma >> 1$, emits radiation which in the laboratory frame is concentrated mostly in the forward direction, in a cone of angle $\theta \sim \gamma^{-1}$. A distant observer in the plane of the orbit of radius R will be within this cone only for a path segment $\delta x \sim 2R/\gamma$. But the pulse of radiation in the

forward direction will be foreshortened by a factor of $1 - \beta = (2\gamma^2)^{-1}$, so he sees a pulse of duration $\delta t = \delta x/2\gamma^2 c = R/\gamma^3 c$. At 80 Mev, R = 29 cm, this gives $\delta t = 2 \times 10^{-16}$ sec, just the radian period $(1/\omega)$ of blue light.

Recently, those of us with mode-locked lasers were proud of having produced picosecond light pulses. In fact, the synchrotron was producing light pulses a thousand times shorter than that, thirty years ago!

In any problem involving the wave equation and the geometry of a circle, Bessel functions are bound to show up; and Schwinger obtained the following quantitative result. An electron in uniform circular motion at orbital frequency ω_o, velocity $v = \beta c = \omega_o R$, radiates power into the n'th harmonic of

$$P_n = \frac{n\omega_o e^2}{R\gamma^2}\left[2\beta^2\gamma^2 \; J'_{2n}(2n\beta) - \int_0^{2n\beta} J_{2n}(x)dx\right] \qquad (1)$$

where, as always, $\gamma \equiv E/mc^2 = (1 - \beta^2)^{-\frac{1}{2}}$. This is the exact formula for a point charge. In the nonrelativistic limit $\beta << 1$, it goes into

$$P_n = \frac{2\omega_o e^2}{R} \; \frac{n+1}{(2n+1)!} \; (n\beta)^{2n+1} \qquad (2)$$

which for $n = 1$ reproduces the elementary textbook result. In the extreme relativistic limit $\gamma >> 1$, various approximate forms exist, depending on how n compares to a critical harmonic number $n_c = 3\gamma^3/2$. For example, if $n << n_c$, we have

$$P_n \simeq \frac{3^{1/6}\Gamma(2/3)\omega_o e^2}{\pi R} n^{1/3} \qquad (3)$$

slowly increasing with n, while if $n >> n_c$,

$$P_n \simeq \frac{\omega_o e^2}{R\gamma} (3n/2\pi n_c)^{1/2} \exp(-n/n_c) \qquad (4)$$

rapidly decreasing with n.

The total power radiated is

$$P = \sum_{n=1}^{\infty} P_n = \frac{2\omega_o e^2}{3R} \beta^3 \gamma^4 \qquad (5)$$

which places the ultimate limit on attainable energy, as noted by Iwanenko and Pomeranchuk; although for the aforementioned synchrotron at 80 Mev it is not yet serious, amounting only to about 12 ev per electron per turn. However, it is only because of the high frequency of the radiation, which keeps the emission from different electrons

incoherent, that the synchrotron will work. If the radiation were of such a low frequency that the bunch of N electrons were smaller than a wavelength, they would radiate coherently, losing energy at a rate proportional to N^2 rather than N; and that would be fatal.

Finally, a controversy, amusing in retrospect, developed for a short time on the validity of classical vs. quantum electrodynamics for this problem. For the summing up, vindicating classical theory, see Schiff (1952).

LINEAR ACCELERATORS

In the closing days of World War II, I found myself in the Navy headed for the South Pacific; but just before leaving I visited my friend Professor W. W. Hansen, at Stanford. Alarmed at my imminent cultural deprivation, he gave me his copy of Whittaker and Watson, with all his marginal comments on the theorems (which I still have and use), to work through while away. This was just managed when, in the Spring of 1946, finally out of the Navy and

having a few months before starting graduate school at Berkeley, I again visited Hansen and proclaimed myself an analyst, thanks to his book.

He immediately put me to work for the Summer, applying all this erudition on the design of the first linear accelerator tubes (about 4 inches in diameter, with loading discs to reduce the phase velocity to c, spaced about one inch apart with central holes about one inch in diameter). First, I had to prove myself by reproducing independently--in whatever way I chose--his calculated results for the dispersion curve of the structure, in its dependence on the dimensions of the loading discs. Having passed that test, he then gave me the problem of developing coupling systems to match the periodic accelerator structure to conventional 3 GHz waveguide. Probably the only part of that work that survives usefully today is the diagnostic method (Jaynes, 1952) worked out to determine, from measurements in the waveguide, when we had a pure running wave in the accelerator tube.

That Summer the news of Schwinger's still unpublished calculations had already reached Stanford (Schiff,

1946), and those of us dreaming of future LINACS felt left out, since it appeared that our machine would not produce this interesting visible light whose color shows, at a glance, the electron energy. But then it occurred to me that, because my coupling systems surely would not work perfectly, giving an absolutely pure running wave in the accelerator tube, the electrons in a real LINAC would, after all, emit some high-frequency radiation.

The "useful" accelerating wave, carrying perhaps 10 megw of power, will be accompanied, inevitably, by a small reflected wave of perhaps 1 megw.

Let us think of it as a plane wave, of a microwave frequency $\omega_o/2\pi = 3$ GHz, Poynting vector $S_o = P_o/A = 2 \times 10^5$ watts/cm^2, traveling in the (-z) direction.

[Of course, the reflected wave in a LINAC is not a plane wave, the transverse field $E_r \sim J_1(kr)$ vanishing on the axis where one tries to keep the electron beam. But by the Poynting theorem, a given reflected power P_o requires a certain average $\langle E_r^2 \rangle$ over the disc apertures A. Therefore, while our plane wave treatment does not describe the situation that LINAC operators now try to achieve, it does describe, within unimportant numerical

factors, a situation that can only too easily be achieved.]

Now consider an electron moving in the (+z) direction with energy $E = \gamma mc^2$. In the electron's reference frame F', this reflected wave appears as one of frequency and Poynting vector

$$\omega' = \gamma(1 + \beta)\omega_0 \simeq 2\gamma\, \omega_0 \quad , \tag{6}$$

$$S' = \gamma^2(1 + \beta)^2\, S_0 \simeq 4\gamma^2\, S_0 \ . \tag{7}$$

For example, at 100 Mev, $\gamma = 1 + (100/0.511) = 197$, the electron sees a plane wave of wavelength $\lambda' = \lambda_0/394$ = 0.025 cm, with an energy flux $S' = (394)^2\, S_0 = 31{,}000$ megawatts/cm^2. From this, it is clear why electron in a LINAC will, by Thomson scattering, emit a considerable amount of high-frequency radiation. And in a wave of this intensity, the electron might oscillate with such amplitude that it generates appreciable radiation in harmonics of the frequency ω'. We then need to discuss Thomson scattering in a wave of very high intensity.

RADIATION FROM AN OSCILLATING CHARGE

To fix numerical magnitudes, an energy flux of the

above value S' corresponds to an electric field strength $E_r \sim 10^4$ esu, and the electron will oscillate with amplitude $a = eE_r/m\omega^2 \sim 10^{-4}$ cm, reaching a peak velocity $v = \omega a \sim 0.024$ c. Even at this intensity, therefore, the scattering as seen in the electron's reference frame F' is not a particularly relativistic problem.

That an oscillating charge will in general radiate harmonics of its orbital frequency is clear from the fourier expansion of the charge distribution;

$$\delta(x - a\cos\omega t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{2^n T_n(x/a)}{\sqrt{a^2 - x^2}}\, e^{-in\omega t} \quad , \quad |x| < a \tag{8}$$

where $T_n(z)$ are the Tchebycheff Polynomials:

$$T_n(z) = \frac{\cos(n\cos^{-1} z)}{2^{n-1}} = T_{-n}(z) \quad . \tag{9}$$

All harmonics are present, but it looks at first glance as if we have escaped from Schwinger's Bessel functions to something new. But of course linear sinusoidal motion is just projected circular motion; and when we calculate the resulting distant field the Bessel functions must inexorably reappear. For example, in Lorentz gauge

the potential ϕ from the retarded solution of $\Box\phi + 4\pi\rho = 0$ is

$$\phi(R) = \frac{e}{2\pi}\sum_n 2^n\, e^{-in\omega t}\int_{-a}^{a} \frac{T_n(x/a)}{\sqrt{a^2 - x^2}}\,\frac{e^{inkr}}{r}\,dx \qquad (10)$$

where $k = \omega/c$, and r is the distance from the point of integration to the point of observation R; a similar expression holds for the vector potential. But when $R \gg a$, these integrals go into the form

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{T_n(z)}{\sqrt{1 - z^2}} e^{iqz}\,dz = (i/2)^n\, J_n(q) \qquad (11)$$

and (1) reduces to

$$\phi(R) = e\sum_n i^{-n}\, J_n(nka\,\cos\alpha)e^{-in\omega t}\,\frac{e^{inkR}}{R} \qquad (12)$$

where α is the angle between the direction of observation and the direction of motion. Already, this looks very much like some of Schwinger's results.

The rest of the solution is straightforward, and we find for the power radiated, in the n'th harmonic, into the element of solid angle $d\Omega$:

$$\frac{dP_n}{d\Omega}\, d\Omega = \frac{e^2\omega^2 n^2}{2\pi c} J_n^2(nka\cos\alpha)\, \tan^2\alpha\, d\Omega \qquad (13)$$

and the total power in the n'th harmonic is

$$P_n = \frac{2e^2 n^2 \omega^2}{c} \int_0^1 J_n^2(nkaq)\,(q^{-2} - 1)\, dq \quad . \qquad (14)$$

These are the exact results for a point charge following the orbit (8). If $nka \ll 1$, (14) goes into

$$P_n = \frac{4e^2 n^2 \omega^2 (nka/2)^{2n}}{c(n!)^2(4n^2 - 1)} \qquad (15)$$

which again, for $n = 1$ yields the elementary textbook result $P_1 = e^2 a^2 \omega^4 / 3c^3$ (half of Schwinger's P_1, since his circular orbit amounts in this limit to two orthogonal dipoles radiating independently).

At other electron energies than 100 Mev, we see from (6), (7) that the transverse displacement, a, will vary as γ^{-1}, while

$$ka = \frac{e}{mc\omega_o}\left(\frac{4\pi S_o}{c}\right)^{1/2} \qquad (16)$$

is independent of γ. In the Stanford LINAC, $ka \sim 0.02$,

and harmonic production is always small. A machine operating at higher power and/or lower frequency would generate correspondingly more harmonics.

BACK TO THE LABORATORY SYSTEM

In view of the foregoing, we need consider only the conventional Thomson scattering at the fundamental frequency ω', in spite of the enormous intensity S' of the incident wave. In the electron's reference frame, it scatters a dipole wave of frequency ω' and intensity given from (13) as

$$\frac{dP_1}{d\Omega}\, d\Omega = \frac{e^2 c}{8\pi a^2} (ka)^4 \sin^2\alpha \, d\Omega \quad . \tag{17}$$

To find how this appears in the laboratory frame, we introduce an azimuth angle ϕ about the z-axis measured from the plane of polarization, and the angle θ' between some propagation direction and the z-axis, as seen in the electron's reference frame. From spherical trigonometry, we can replace in (17)

$$\sin^2\alpha = 1 - \cos^2\phi \sin^2\theta' \quad . \tag{18}$$

Applying the relativistic abberation and Doppler effect formulas, radiation which in the electron's reference frame is emitted at an angle θ' to the z-axis with frequency ω', will appear in the laboratory frame to have direction θ, frequency ω, where

$$\cos\theta = \frac{\beta + \cos\theta'}{1 + \beta\cos\theta'} \quad , \tag{19}$$

$$\omega = \gamma(1 + \beta\cos\theta')\omega' = 2\gamma^2(1 + \beta\cos\theta')\omega_o \quad . \tag{20}$$

In the electron's frame half the energy is emitted in the forward hemisphere $0 < \cos\theta' \leq 1$. This appears in the laboratory frame as $\beta < \cos\theta \leq 1$, which becomes in the extreme relativistic limit $0 \leq \gamma\theta < 1$, just the aforementioned forward cone of angle $\theta = \gamma^{-1}$. Since both energy and frequency of a plane wave transform as the time component of a four-vector, the ratio $n \equiv E/\hbar\omega$ is an invariant; i.e., half the "photons" are emitted into this cone.

Eliminating θ' between (19) and (20), we have the variation of frequency with angle as seen in the laboratory system:

$$\omega = \frac{(1+\beta)\omega_o}{1-\beta\cos\theta} \simeq \frac{4\gamma^2\omega_o}{1+\gamma^2\theta^2} \quad . \tag{21}$$

The observed frequency is a maximum in the forward direction, and drops to half that value at the cone angle $\theta = \gamma^{-1}$.

At 100 Mev, with ω_o corresponding to a microwave wavelength $\lambda_o = 10$ cm, the wavelength at the center of this spot would be $\lambda = \lambda_o/4\gamma^2 = 6460$ Å, or visible red light. At 127 Mev, we would have at the center of the spot $\lambda = 4000$ Å, about the limit of visibility in the blue; and this would shade through the spectrum to red at an angle of somewhat less than $\theta = \gamma^{-1} = 0.23$ degrees from the axis. At any higher electron energy, the visible radiation would appear in a tiny circular rainbow, blue on the inside to red on the outside, whose cone angle would tells us the electron energy.

To find the absolute intensity and spectral distribution of energy as seen in the laboratory system, we can take advantage of the invariance of photon number (a useful mathematical property whether or not one has a literal belief in photons). Suppose the accelerator pulse has duration $t_o \sim 2 \times 10^{-6}$ sec in the laboratory

system. In the electron's frame this will be contracted to $t' = t_o/2\gamma$, during which time it is scattering the power $P_1 = e^2c(ka)^4/3a^2$. Crudely, then, we can say that during a pulse an electron shakes off about

$$N = \frac{P_1 t'}{\hbar\omega'} = \frac{e^2}{\hbar c}(ka)^2 \frac{\omega_o t_o}{3} \tag{22}$$

photons. Being invariant, this same number will appear in the laboratory system, transformed to various frequencies. [In fact, as we see from (16), N is not only Lorentz invariant, it is also independent of the electron energy γ, being determined by the value of $(S_o t_o/\omega_o)$ in the machine design; i.e., on the number of microwave photons supplied per pulse.]

Integrating (18) over the azimuth angle ϕ, the number of photons emitted into the solid angle $d\Omega' = 2\pi\, d\cos\theta'$ of an annular ring $d\theta'$, is

$$dN = \frac{3N}{4}(1+\cos^2\theta')\, d\cos\theta' \tag{23}$$

In the laboratory frame, these same photons will appear in the annulus of solid angle $d\Omega = 2\pi\, d\cos\theta \simeq 2\pi\theta\, d\theta$. From (21) they have frequencies in the range given by

$$\frac{d\omega}{\omega} = -\frac{2\gamma^2 \theta\, d\theta}{(1+\gamma^2\theta^2)} \tag{24}$$

while in the relativistic limit, the abberation law (19) goes into

$$\cos\theta' = \frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2} \quad . \tag{25}$$

In the laboratory frame, therefore, we obtain an energy in the range $d\omega$ of $I(\omega)d\omega = \hbar\omega\, dN$; i.e., the spectral energy density is from (23)-(25):

$$I(\omega) = \frac{\hbar\omega dN}{d\omega} = 3\hbar N\left(\frac{\omega}{\omega_{max}}\right)^3\left[1+\left(\frac{\omega_{max}}{\omega}-1\right)^2\right] \tag{26}$$

where from (21), $\omega_{max} = 4\gamma^2\omega_o$ is the highest frequency obtained.

The light energy per electron per pulse that is emitted into the forward cone $\theta = \gamma^{-1}$ is then

$$\int_{\frac{1}{2}\omega_{max}}^{\omega_{max}} I(\omega)\, d\omega = \frac{25}{32} N\hbar\omega_{max} \quad . \tag{27}$$

From (22) and the numerical values discussed above, we find $N \simeq 0.04$. If at each pulse the electron gun injects

100 milliamperes of current for one microsecond, it would provide 6×10^{11} electrons. Taking 10^{11} as a conservative value, we then find that the machine should emit, at each pulse, a light energy of about $3 \times 10^{9}\ \hbar\omega_{max}$, an easily visible amount. This is only a crude estimate, ignoring the change of electron energy during a pulse. Clearly, however, a device made specifically to enhance this radiation could produce far more than our estimate. Thus, while the LINAC does not provide the spectacular light radiation of the synchrotron, it should produce some radiation that becomes visible above 100 Mev.

Of course, the fact that $N \ll 1$ above is irrelevant to the argument, which actually used only the fact that the the ratio (energy)/(frequency) is Lorentz invariant; and this is true whether or not photons are physically real. Indeed, the term "photon" may be construed merely as a convenient Lorentz-invariant unit of energy; just as "magneton" denotes not a physical particle, but a certain amount of magnetic moment. Then there can be no objection to use of the photon in purely classical calculations. To emphasize this, note that our final result (27) is independent of the numerical value of $\hbar$, because $N\hbar$ is.

THE UNDULATOR

In the Summer of 1947 I returned to Stanford and wrote up the above analysis, which was circulated privately to interested persons, but was not considered of enough earth-shaking importance to warrant publication. Clearly, as efficient sources of visible light, synchrotrons and LINACS compare rather unfavorably with Thomas Edison's tungsten filament. The idea of producing coherent visible light was at that time far beyond the dreams of science fiction; probably, if anyone had dared to suggest it, physicists would have held such a thing to be fundamentally impossible for one of those mysterious quantum-mechanical reasons that everybody invokes but nobody understands.

However, with the rapid development of molecular spectroscopy and thoughts of ultra-short-pulse radars, efforts to push the usable microwave spectrum into the millimeter-wave region were pursued vigorously in many laboratories. My prediction of visible light from the LINAC came to the attention of Hans Motz, and he conceived a variant of the idea which could produce coherent millimeter waves from a high-energy electron beam.

The Undulator (Motz, 1950) passes the electrons through a comb-like structure of permanent magnets, reversing the magnetic field at each tooth of the comb. Thus the electron beam is forced to "undulate" and radiate light which, in the forward direction, can be relatively intense because at these longer wavelengths a bunch of N electrons can be made to radiate coherently. Needless to say, the Undulator worked as predicted, but with the large amount of apparatus needed, it was hardly competitive with other methods of generating millimeter waves. To the best of my knowledge, no further work has been done on it since the early 1950's.

THE CYCLOTRON-MASER EQUATIONS

Also in the Summer of 1947, having just completed J. R. Oppenheimer's course in quantum theory at Berkeley, I was toying with alternative ways of writing the Schrödinger equation. One form in particular made an impression on me, although it is simple only for a two-level system. Suppose a two-level atom is perturbed by an electric field: $V = -\mu E(t)$ where μ is the dipole moment

matrix element between the levels. Then writing $\psi(t)$ $= a_1(t)\psi_1 + a_2(t)\psi_2$, the amplitudes satisfy

$$i\hbar\dot{a}_1 = E_1 a_1 - \mu E(t) a_2 \quad , \tag{28a}$$

$$i\hbar\dot{a}_2 = E_2 a_2 - \mu E(t) a_1 \quad . \tag{28b}$$

If we introduce the quantities

$$W(t) = E_1 |a_1|^2 + E_2 |a_2|^2 - \tfrac{1}{2}(E_1 + E_2) \tag{29}$$

$$M(t) = 2\mu \mathrm{Re}(a_1 a_2^*) \tag{30}$$

usually interpreted as expectations of energy and dipole moment, then Equations (28) can be rewritten in the suggestive form

$$\ddot{M} + \omega^2 M = -K^2 W E(t) \tag{31a}$$

$$\dot{W} = E(t)\dot{M} \tag{31b}$$

where $\omega = (E_2 - E_1)/\hbar$, $K = 2\mu/\hbar$. We see that the Schrödinger equation has a simple physical content that cannot be seen from (28); it states that the dipole moment oscillates according to a driven harmonic oscillator equation, the only "new" feature being that the coupling constant $(-K^2 W)$ varies slowly with time and has opposite signs depending on whether the atom is near the upper or lower state.

That a two-level system can be given a Bloch sphere representation is seen at once from the fact that Equations (31) possess the first integral

$$\dot{M}^2 + \omega^2 M^2 + K^2 W^2 = \text{const} = \omega^2 \mu^2 \qquad (32)$$

which, in suitable coordinates (x,y,z) proportional to $(\dot{M},M,W)$, is the unit sphere, $x^2 + y^2 + z^2 = 1$.

When, in the middle 1950's, Professor Willis Lamb showed me his theory of the ammonia maser, I immediately rewrote it in the form (31), which made the physical operation of the device seem much clearer. This marked the beginning of neoclassical theory (Jaynes, 1973).

Bear with me--we have not really left the topic of free electrons. For also in the middle 1950's, Professor Charles Kittel gave a Seminar talk at Stanford on cyclotron resonance techniques for elucidating band structure in solids. This set me to thinking again about the generation of radiation from fast electrons, using static magnetic fields in other ways than in the undulator.

The most obvious thing is just the inverse cyclotron; fire electrons into a uniform magnetic field and they will radiate at their cyclotron frequency.

Although that idea has been pretty well exploited and extended in the magnetrons that today cook our food, let us look at the equation of motion of a nonrelativistic electron

$$m\dot{v} = e\left[E + \frac{1}{c}(v \times H)\right] \tag{33}$$

in which $H_z = H_o = \text{const.}$, while other field components may be time dependent. If we have only oscillating electric fields, $H_x = H_y = 0$, (33) yields

$$\dot{v}_x - \omega_o v_y = \frac{e}{m} E_x \tag{34a}$$

$$\dot{v}_y + \omega_o v_x = \frac{e}{m} E_y \tag{34b}$$

where $\omega_o \equiv (e/mc)H_o$ is the cyclotron frequency. If $E_y = 0$, we can write $\ddot{y} = \dot{v}_y = -\omega_o v_x = -\omega_o \dot{x}$, whence by a time integration and suitable choice of the origin, (34a) becomes a driven harmonic oscillator equation for the x coordinate of the electron:

$$\ddot{x} + \omega_o^2 x = \frac{e}{m} E_x(t) \quad . \tag{35}$$

This is the usual result, and is not of particular interest for present purposes; the coupling constant is invariable and so the phenomena allowed by (31) in the maser do not appear.

Now let's introduce a variable magnetic field $H_x(t)$. If H_y and the electric field are zero, the equations of motion are

$$\dot{v}_x - \omega_o v_y = 0 \tag{36a}$$

$$\dot{v}_y + \omega_o v_x = \frac{e}{mc} v_z H_x \tag{36b}$$

$$\dot{v}_z = - \frac{e}{mc} v_y H_x \quad . \tag{36c}$$

By a time integration of (36a) and proper choice of the origin we have $v_x = \omega_o y$; and the equations of motion reduce to

$$\ddot{y} + \omega_o^2 y = \frac{e}{mc} H_x v_z \tag{37a}$$

$$\dot{v}_z = - \frac{e}{mc} H_x \dot{y} \quad . \tag{37b}$$

It is apparent at once that these have the same structure as (31); but let us make the analogy stronger by defining the y-component of electric dipole moment, and a quantity of the dimensions of the energy:

$$M(t) = ey(t) \tag{38}$$

$$W(t) = -mc\, v_z(t) \tag{39}$$

Equations (37) then reduce to

$$\ddot{M} + \omega_o^2 M = -K^2 W H_x(t) \tag{40a}$$

$$\dot{W} = H_x(t) \dot{M} \tag{40b}$$

with $K = (e/mc)$.

The formal analogy with the two-level Schrödinger equation (31) is complete. Of course, the numerical constants and physical meanings are different. For example, in the case of the ammonia molecule in a maser, the right-hand side of (32) is fixed at $\omega^2\mu^2$ by laws of physics [in fact, (32) is just the relation $|a_1|^2 + |a_2|^2 = 1$ usually called conservation of probability]; while the corresponding integral of (40) merely expresses conservation of energy, and the right-hand side can have any value, determined by the initial conditions. Also, a magnetic field alone cannot transfer energy to or from the electron, since the Lorentz ($v \times H$) force is orthogonal to the velocity. So we must introduce also electric field terms into the equations of motion before we can obtain a maser-like amplification.

Suppose, then, that the above field H_x is part of a plane wave traveling in the z-direction, so that we have a field $E_y = -\alpha H_x$. [In free space $\alpha = 1$, but in a guided wave structure α measures the wave impedance, and can be either greater or less than unity.] We must then add a term $(e^2/m)E_y$ to the right-hand side of (40a), and it becomes

$$\ddot{M} + \omega_o^2 M = \frac{e^2}{mc}(v_z - \alpha c)\, H_x \quad . \tag{41}$$

Suppose that initially an electron is injected at velocity v_z along the z axis, so that $M(0) = \dot{M}(0) = 0$, and a wave $H_x = H_1 \cos(\omega t - kz + \phi)$ is present. Let us solve (41) for a time interval many cycles of ω_o but short enough so that v_z has not changed appreciably. At time t, the electron then sees the field at position $z = v_z t$, so its transverse motion is given by

$$\ddot{M} + \omega_o^2 M = A \cos(\omega' t + \phi) \tag{42}$$

where

$$\omega' \equiv \omega - k v_z \quad , \tag{43}$$

$$A \equiv \frac{e^2}{mc}(v_z - \alpha c)\, H_1 \quad . \tag{44}$$

The solution is

$$\dot{M}(t) = A \int_o^t \cos\omega_o(t - t')\cos(\omega' t' + \phi)\, dt'$$

$$= \frac{A}{2} \frac{\sin(\omega' t + \phi) - \sin(\omega_o t + \phi)}{(\omega' - \omega_o)} \tag{45}$$

and at time t the electron has transferred energy to the field of

$$E(t) = -\int_o^t E_y(t')\, \dot{M}(t')\, dt \tag{46}$$

$$= \frac{\alpha A H_1}{4} \left[\frac{1 - \cos(\omega' - \omega_o)t}{(\omega' - \omega_o)^2} \right] \qquad (46)$$

independent of the initial phase ϕ. This is a maximum at resonance, $\omega' = \omega_o$, for which

$$E(t) = \frac{e^2 H_1^2}{8mc} (v_z - \alpha c)\, t^2 \qquad (47)$$

and the criterion for amplification is

$$v_z > \alpha c \quad . \qquad (48)$$

We then need $\alpha < 1$, which is the case for a TM waveguide mode.

From waveguide theory, the wave impedance and propagation constant for a TM wave mode of cutoff frequency ω_c are given by

$$ck = \omega\alpha = (\omega^2 - \omega_c^2)^{\frac{1}{2}} \quad . \qquad (49)$$

The system will therefore amplify and oscillate, delivering the kinetic energy $mv_z^2/2$ to the radiation field, at a frequency ω given by the root of

$$\omega - \frac{v_z}{c}\sqrt{\omega^2 - \omega_c^2} = \omega_o \qquad (50)$$

provided that (48) is also satisfied at this value of ω.

By graphical analysis one sees that these conditions are readily satisfied; for example, choosing $\alpha = 1/4$, $v_z/c = 1/2$, we find a solution for

$$\omega = \frac{8}{7}\,\omega_o = 1.143\,\omega_o \tag{51}$$

$$\omega_c = \frac{2}{7}\sqrt{15}\,\omega_o = 1.107\,\omega_o \tag{52}$$

The stimulation from Kittel's talk carried me this far by late 1959. Clearly, much more analysis is needed, to calculate gain, power, efficiency, etc. before one would be in a position to pass judgment on the practicality of such a system. It is interesting that the possibility of wave amplification arises here from the sign reversal of coupling coefficient in (41) at $v_z = \alpha c$, just as the possibility of amplification in a maser or laser arises from the sign reversal of W in (31) that signifies the change from net absorption to net stimulated emission.

Needless to say, this analysis leaves it an open question whether other configurations might be superior in power, efficiency, stability, etc. Presumably, use of a circularly polarized wave would double the rate of energy transfer.

SCHWARZ—HORA EFFECT (?)

An entirely different principle by which free electrons might be made to produce light, about which I and many other people did some thinking in 1970, concerns the "blue electron" effect reported by Schwarz and Hora (1969). I don't want to get into controversies about whether the effect has in fact been seen; and whether it could have been, with present electron optics techniques. But I think it is legitimate and important to raise the question whether, in principle, according to present quantum theory, such an effect should exist.

In fact, a number of theoreticians (including myself) were able, without the slightest difficulty, to give quantum-mechanical calculations which predicted virtually everything that Schwarz and Hora reported seeing. And, at least in my own version of the theory, there was no reason to think that technical problems of collimation would prevent one from seeing this light.

It does not speak well for quantum theory if it can so glibly account for nonexistent effects. Anyone who believes that this has happened can hardly avoid asking how many other nonexistent effects it has been predicting

all these years; and how much we have been misled in our picture of Nature's workings, as a result.

Schwarz and Hora reported that 50 kev electrons ($v/c \sim 0.4$) were diffracted by a crystal and simultaneously irradiated by blue light (4880 Å) from an argon laser, polarized with the electric vector parallel to the electron beam; and that on drifting 25 cm and striking an alumina screen (nonluminescent or nearly so for ordinary electrons), they emitted blue light at the position of the normal diffraction spots.

Many people immediately took up a simple classical explanation: the light produces a periodic velocity modulation, making the arrangement an "optical klystron." However, the reported primary beam current (10^{-6} ampere) was only one electron per 100 optical cycles; the current falling on any one diffraction spot must have been far less. Thus there are no real "bunches." Under these conditions, one should have observed instead an incoherent continuous spectrum of light, with only a negligible portion appearing at the optical frequency ω.

To demonstrate this, let us ignore for the moment the difficulties about velocity spread and suppose perfect

coherence; electrons can arrive at the screen only at the instants $t_k = k\tau$, where $\tau \equiv 2\pi/\omega$ is the optical period, and let n_k be the number of electrons arriving at time t_k. If the average current is n electrons per second, the n_k are random variables with expectations $\langle n_k \rangle = n\tau$, and the net current is then the random function

$$J(t) = e \sum_{k=-\infty}^{\infty} n_k \, \delta(t - t_k) \; . \tag{53}$$

With independent Poisson distributions for the n_k, the second moments are $\langle n_k n_{k'} \rangle = n^2\tau^2 + n\tau\delta_{kk'}$, which yields the spectral density

$$I(\nu) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \int_{-T}^{T} dt' \langle J(t)J(t') \rangle e^{i\nu(t-t')}$$

$$= e^2 \left[n + 2\pi n^2 \sum_{r=-\infty}^{\infty} \delta(\nu - r\omega) \right] \tag{54}$$

of which the first term was given long ago by Schottky. The radiation from this current thus consists of a continuous spectrum of amplitude proportional to n, superimposed on a line spectrum at the fundamental and its harmonics, proportional to n^2. The ratio (energy in line spectrum)/(energy in continuous spectrum) is from (54) $2\pi n^2/n\omega = n\tau$, just the average number of electrons per bunch.

In a microwave klystron, $n\tau \gg 1$, and we obtain essentially a pure line spectrum; but in the SH experiment, $n\tau < 10^{-3}$ for any diffraction spot, so according to classical bunching theory practically all the radiation should be in the continuous spectrum. With imperfect coherence, the line spectrum will be still further suppressed.

We conclude that the effect cannot depend on any regularity in arrival times of different electrons; instead, each individual electron must, in some way, retain a "memory" of the light frequency for at least a million cycles after irradiation, and therefore the effect must be an essentially quantum-mechanical one.

We indicate such a mechanism by a crude calculation. An incident electron is represented by a wave packet

$$\Psi_o(x,t) = (2\pi)^{-3}\int d^3k\ \Psi_o(k,t)\exp(ik\cdot x) \qquad (55)$$

where

$$\Psi_o(k,t) = \Psi_o(k)\exp(-i\omega_k t) \qquad (56)$$

with

$$\omega_k = \hbar k^2/2m \quad , \qquad (57)$$

and the light wave by the vector potential

$$A(x,t) = A_o\varepsilon\cos(\omega t - q\cdot x + \phi) \qquad (58)$$

where ε is a unit polarization vector, $\varepsilon \cdot q = 0$, and $\omega = cq$. Since the electron momentum is well defined, $\Psi_0(k)$ will be sharply peaked about the value k_0 such that $\hbar^2 k_0^2 = 2mE$, and approximations of the form $(\varepsilon \cdot k_0) \simeq (\varepsilon \cdot k)$ will be justified.

The perturbed wave packet is $\Psi(x,t) = \Psi_0 + \Psi_1 + \ldots$, where the first-order term is the solution of

$$i\hbar\dot{\Psi}_1 - p^2/2m\ \Psi_1 = -(e/mc)\ A \cdot p\ \Psi_0(x,t) \qquad (59)$$

which vanishes in the remote past. In the experiment, the electrons drifted through a narrow light beam, spending of the order of 40 to 400 optical cycles in it. Accordingly, suppose that the field (58) is switched on at time $t = 0$, and off again at $t = \tau$ (actually, the sharp edges of the crystal provide this switching, which prevents the solution from following adiabatically in the fringing field at the edge of the light beam). The solution of (59) for $t > \tau$ is then

$$\Psi_1(k,t) = \frac{eA_0}{2mc}\ (\varepsilon \cdot k)\left[e^{i\phi}\ \Psi_0(k+q)\ \frac{e^{i\omega_+ \tau} - 1}{\omega_+} + e^{-i\phi}\ \Psi_0(k-q)\ \frac{e^{i\omega_- \tau} - 1}{\omega_-} \right] e^{-i\omega_k t} \qquad (60)$$

where $\omega_\pm = \omega_k - \omega_{k\pm q} \pm \omega$. From this one sees that there is

no secular buildup of either term, since from the exact dispersion law, $\omega_k = c[k^2 + (mc/\hbar)^2]^{\frac{1}{2}}$, ω_+ and ω_- cannot vanish for any value of k (in diagram language, energy and momentum cannot both be conserved at a single vertex). Furthermore, in the experimental arrangement $q \cdot k_o = 0$, and $\omega_q/\omega = \hbar q/2mc \simeq 10^{-6}$, so the differences $|\omega_k - \omega_{k\pm q}|$ are negligible compared to ω for all k values for which $\Psi_o(k \pm q)$ is appreciable. Therefore, we may write simply $\omega_\pm = \pm\omega$.

Presence of the amplitude factor $(\varepsilon \cdot k) \simeq (\varepsilon \cdot k_o)$ in (60) accounts for the reported polarization effect. However, because of the lack of any secular buildup, the perturbation does not tend to emphasize any particular frequency component in Ψ_1, and so memory of the light frequency is not contained in the subsequent time variation of the wave function.

Equation (60) predicts that the amplitude of the effect should oscillate periodically with the interaction time τ, and therefore with the beam voltage, according to $|e^{i\omega\tau} - 1|$. In the following we assume the optimum interaction time, corresponding to $\exp(i\omega\tau) = -1$, and the optimum polarization, $(\varepsilon \cdot k_o) = k_o$.

Noting that the factor $r_o \equiv eA_o/mc\omega$ is just the displacement of a classical point electron in the field A_o and transforming back to ordinary space, the first-order change in the wave function becomes

$$\Psi_1(x,t) = k_o r_o \left[e^{i(q\cdot x-\phi)} \Psi_o(x - vt,t) - e^{-i(q\cdot x-\phi)} \Psi_o(x + vt,t) \right] e^{-i\omega_q t} \qquad (61)$$

where $v \equiv \hbar q/m = 1.5 \times 10^5$ cm sec^{-1} is the recoil velocity of an electron which has absorbed the momentum $\hbar q$ from the light wave. The frequency $\omega_q = \hbar q^2/2m$ is in the microwave range, 1.53 GHz. In first order, therefore, two new wave packets appear which separate from the parent at the recoil velocity, their centers reaching a separation of 3 microns and the phase term $\exp(-i\omega_q t)$ oscillating through about three cycles, during the 2 nsec drift time.

It is one of the most interesting features of this experiment that nothing in the experimental conditions seems to determine the size or shape of the incident wave packet $\Psi_o(x,t)$; and yet, as we will see, observable effects depend on these details. The dimensions of a reasonable wave packet are very large compared to the electron's de Broglie wavelength (4×10^{-10}cm), and the "quantum-mechanical

spreading" of the packets will be negligible; thus we may approximate the incident wave packet by an envelope function $f(x,y,z)$ of constant size and shape. We readily verify that if $\omega_o = \hbar k_o^2/2m$, and $v_o \equiv \hbar k_o/m$, the group velocity, the function

$$\Psi_o(x,y,z,t) \equiv f(x,y,z - v_o t)\ \exp(ik_o z - i\omega_o t) \qquad (62)$$

is an accurate solution of the free-space Schrödinger equation if $\nabla^2 f$ can be neglected in comparison with $k_o^2 f$. In the present case, any reasonably smooth envelope function $f(x,y,z)$ of a few microns width leads to $|\nabla^2 f/k_o^2 f| \sim 10^{-12}$. We have chosen the hyperbolic secant function, taken for ease of calculation in product form:

$$f(x,y,z) = (8a^3)^{-\frac{1}{2}}\ \mathrm{sech}(x/a)\ \mathrm{sech}(y/a)\ \mathrm{sech}(z/a)\ . \qquad (63)$$

The choice $a = 1.70$ microns yields a wave packet of width 3 microns between "half-way down" points of $|\Psi|^2$.

After irradiation, the perturbed wave function corresponds to an effective charge density

$$\begin{aligned} \rho &= e|\Psi_o + \Psi_1 + \ldots|^2 \\ &= e|\Psi_o|^2 + 2e\ \mathrm{Re}(\Psi_o^* \Psi_1) + \ldots \end{aligned} \qquad (64)$$

in which the second term, arising from interference between

the parent and daughter wave packets, contains terms oscillating in space with the wavelength of the light: from (61), (62), and (63) we have

$$2e\ \mathrm{Re}(\Psi_o^* \Psi_1) = \frac{ek_o r_o}{4a^3}[g_-(x) - g_+(x)]\mathrm{sech}^2(y/a)\mathrm{sech}^2(z-v_o t/a) \tag{65}$$

where

$$g_\pm(x) \equiv \cos[q(x \pm vt/2) - \phi]\ \mathrm{sech}(x/a)\ \mathrm{sech}[(x \pm vt)/a] \tag{66}$$

in which we have used the relation $\omega_q = qv/2$ and chosen the x-axis along the direction of the light propagation vector q.

Now the meaty part of the calculation begins. We use the charge density (64) next to find the corresponding z-component of the current $J_z(x,t) = \rho(x,t)\, v_o$. To calculate the radiation we need the fourier transform

$$J_z(k,\omega) = \int_{z<d} d^3x \int dt\ J_z(x,t)\ e^{i(k\cdot x - \omega t)}, \tag{67}$$

the constraint $z < d$ expressing the fact that the current is suddenly decelerated to zero at the screen, which we suppose to be the source of the radiation. The spectral density $I(\omega)$ of the radiation is proportional to $|J_\perp(k,\omega)|^2$, where $J_\perp$ is the component of (67) transverse to the

direction of observation. For radiation in the x-direction we find the analytical result

$$I(\omega) \sim \left| \frac{\sin\left(\frac{kvt}{2}\right) \sin\left[(k-q)\frac{vt}{2}\right]}{\sinh\left(\frac{vt}{a}\right) \sinh\left[\frac{\pi(k-q)a}{2}\right]} \right|^2 . \qquad (68)$$

This yields an asymmetric spectrum in general; computer evaluations of (68) show smooth bell-shaped curves centered at values of (k-q)a usually in the range ±1, with full width at half maximum of about $\Delta k = a^{-1}$; thus in this version of the theory the dimensions of the original wave packet are determined crudely by the observed spectral width. The shape of the spectrum is a kind of convolution of the shape of the original wave packet. Such a measurement, if successful, would represent the first observation of a new property of free electrons, that has never before been accessible to experiment. A "periodicity" like that reported by Schwarz also appears if we examine (68) over a wide range of (vt/a).

In this analysis the effect is contained, not in the time behavior of the wave function, but in its space

periodicity which develops in the drift time following irradiation. Since each electron radiates independently of the others, there should be no particular collimation problems.

Since 1972 there has been a general disinclination to believe in the existence of this effect. While I cannot judge the reliability of an experiment that I did not witness, it does seem to me that definite proof of its nonexistence would be a considerable embarrassment to quantum theory (or at least to the quantum theorists who found it so easy to account for). And against whatever improbability it may have, one must balance the obvious importance of finding out about it, if it does exist. So I hope experimentalists will not forget about it entirely.

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